25 Years of SINGULAR

talk at the

AIMS

-

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SINGULAR

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A Computer Algebra System for Polynomial Computations

with special emphasize on the needs of algebraic geometry, commutative algebra, and singularity theory

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W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann
Technische Universität Kaiserslautern
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The computer is not the philosopher's stone but the philosopher's whetstone

Hugo Battus, Rekenen op taal 1983

Birth of SINGULAR

- 1984

History

1983

Greuel/Pfister: Exist singularities (not quasi-homogeneous and complete intersection) with exact Poincaré-complex?

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Neuendorf/Pfister: Implementation of the Gröbner basis algorithm in basic at ZX-Spectrum

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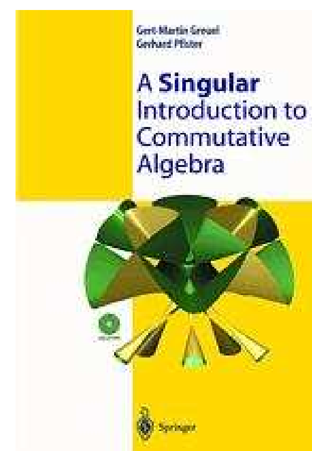

2002

Book: A SINGULAR Introduction to Commutative Algebra (G.-M. Greuel and G. Pfister, with contributions by O. Bachmann, C. Lossen and H. Schönemann).

History

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Jenks Price

for:

Excellence in Software Engineering

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- Supported by: Deutsche Forschungsgemeinschaft, Stiftung Rheinland-Pfalz für Innovation, Volkswagen Stiftung
- SINGULAR is free software (Gnu Public Licence)

Team

T. Wichmann, C. Lossen, G.-M. Greuel, H. Schönemann, W. Pohl, G. Pfister, V. Levandovskyy, E. Westenberger, A. Frühbis-Krüger, Oscar, K. Krüger

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Kaiserslautern
Saarbrücken
Cottbus
Berlin
Mainz
Dortmund
Valladolid
La Laguna
Buenos Aires

surfaces: http://www.imaginary2008.de/

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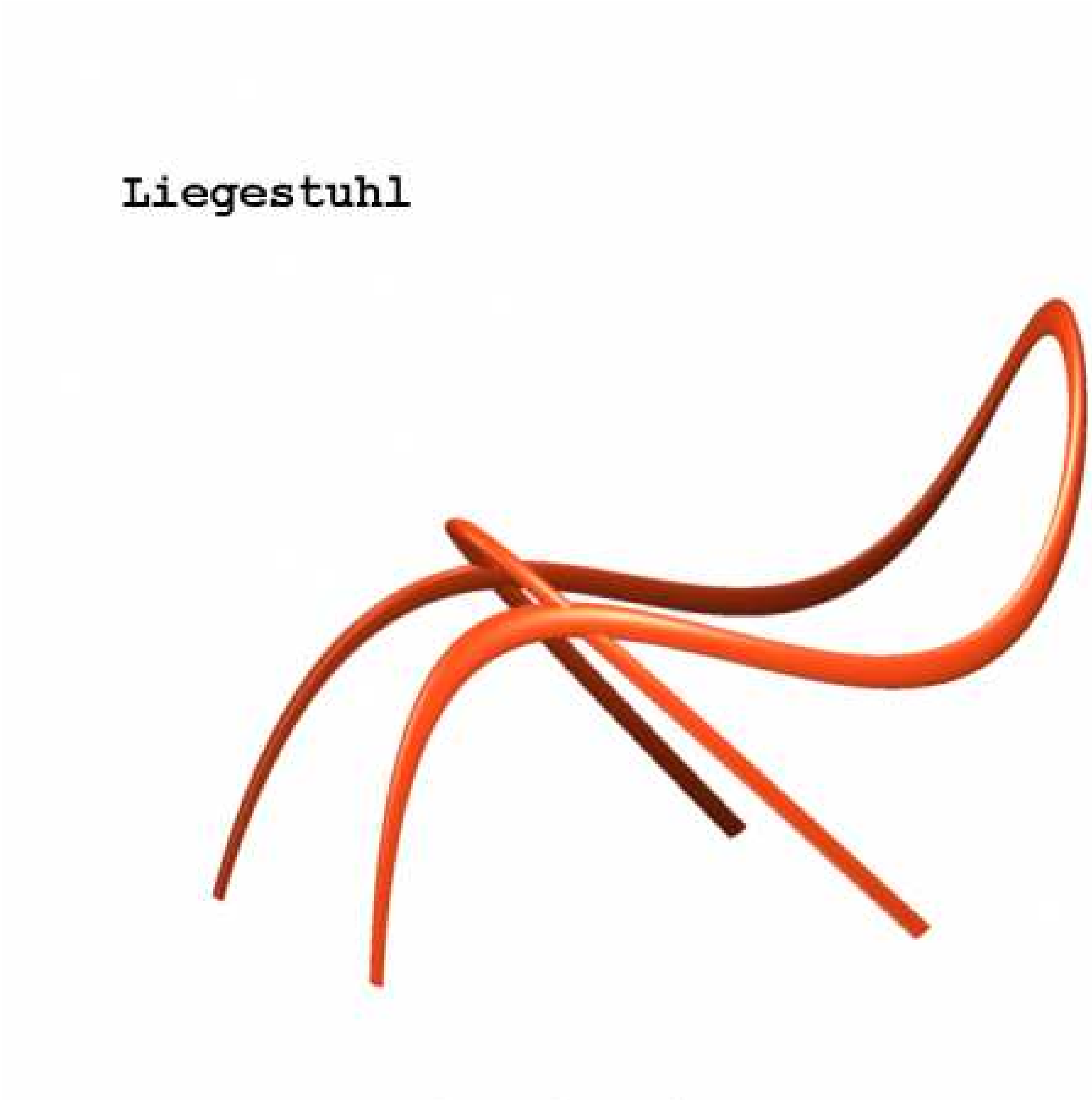
- Consider the equation

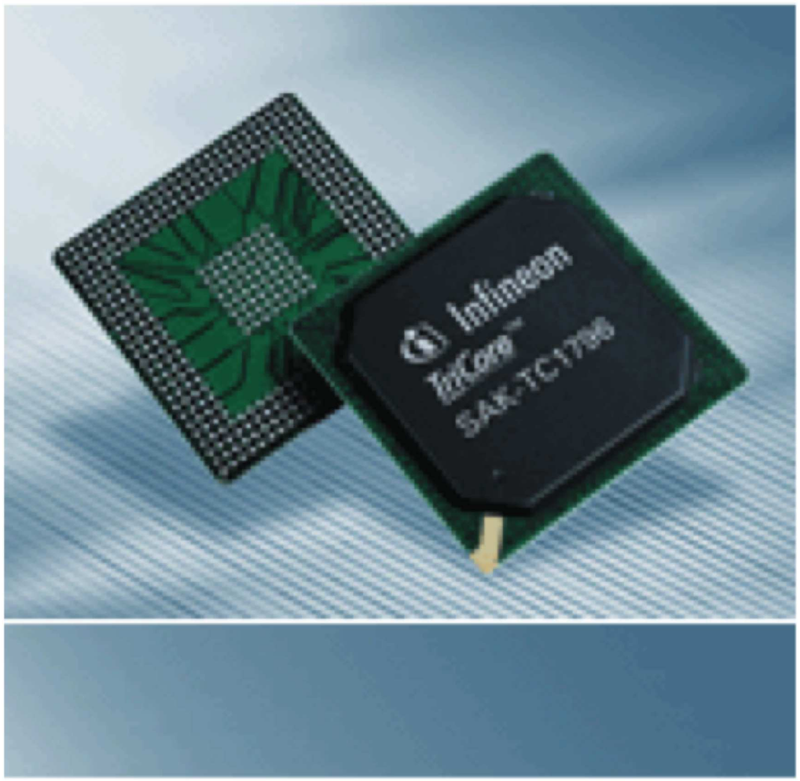
$$((y-z)^2+(y-1-\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3)^2-\tfrac{1}{50})((x-\tfrac{3}{5}y+1)^2+(y-z^2)^2-\tfrac{1}{100})=0$$

- defining a surface of degree 10 in $\mathbb{R}^3$.
- What do you expect from the real picture?

Nice algebraic surfaces

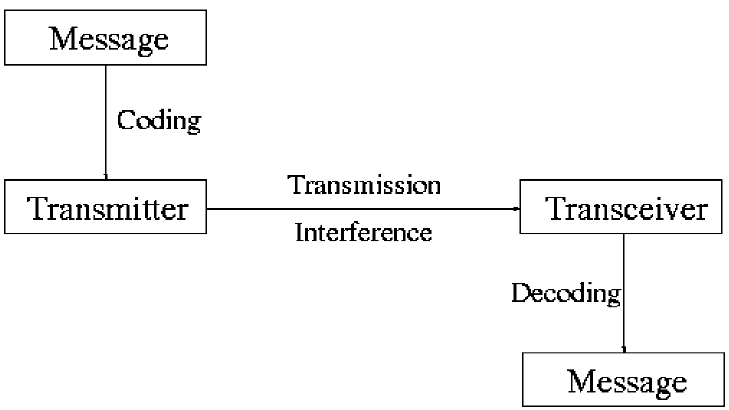


Infineon Tricore Project



- Aim: prove that the processor (32 Bit) works correctly
- every instruction of the processor will be verified specifying special properties and proving them
- it is difficult to check the arithmetic properties

Coding theorie



Coding theory: brnoeth.lib

- AG-Codes are codes, using algebraic vector spaces of divisors on algebraic curves over finite fields
- an implementation of the Brill-Noether algorithm for solving the Riemann-Roch problem and applications in Algebraic Geometry codes is implemented in SINGULAR.

Models for economy

Felix Kubler and Karl Schmedders (University of Zürich)

General problem:

- Study a computer model of a national economy,
  a standard exchange economy with finitely many agents and goods
- especially study equilibria
  Walrasian equilibrium consists of prices and choices, such that household maximize utilities, firms maximize profits and markets clear

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Mathematical problem:
Find the positive real roots of a given system of polynomial equations

equilibrium model with production

```
ring R = 0,x(1..22),dp;
ideal I = -1+x(1)^5*x(4)*x(13),      -1+x(2)^5*x(4)*x(14),
-1+x(3)^5*x(4)*x(15),     -1+x(5)^3*x(8)*x(13),        -1+x(6)^3*x(8)*x(14),
-1+x(7)^3*x(8)*x(15),                              -1+x(9)^4*x(12)*x(13),
-1+x(10)^4*x(12)*x(14),                          -1+x(11)^4*x(12)*x(15),
5+2*x(16)-x(1)*x(13)-x(2)*x(14)-x(3)*x(15),
3+5*x(16)-x(5)*x(13)-x(6)*x(14)-x(7)*x(15),
(x(1)+x(5)+x(9))^3-x(17)^2*x(18),
(x(2)+x(6)+x(10))^2-x(19)*x(20),
(x(3)+x(7)+x(11))^2-4*x(21)*x(22),
x(17)+x(19)+x(21)-10,                              x(18)+x(20)+x(22)-10,
8*x(13)^3*x(18)-27*x(16)^3*x(17),    x(13)^3*x(17)^2-27*x(18)^2,
x(14)^2*x(20)-4*x(16)^2*x(19),          x(14)^2*x(19)-4*x(20),
x(15)^2*x(22)-x(16)^2*x(21),              x(15)^2*x(21)-x(22);
```

Sudoku

				5			8	
				6	2			5
6			4			7		
		7				9	6	
		5	2		6	1		
	3	6				4		
		3			7			4
1			5	8				
	6			1				

Sudoku

- the idea of a Sudoku goes back to Leonard Euler: Latin squares
- in our days invented by Howard Garns (USA): Number place

Sudoku

- the idea of a Sudoku goes back to Leonard Euler: Latin squares
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- associate to the places in a Sudoku the variables $x_1, \dots, x_{81}$ and to each variable x_i the polynomial $F_i(x_i) = \prod_{j=1}^{9}(x_i - j)$
- Let $E = \{(i,j), i < j$ and i,j in the same row, column or $3 \times 3 -$ box$\}$
- For $(i,j) \in E$ let $G_{i,j} = \frac{F_i - F_j}{x_i - x_j}$.
- Let $I \subset \mathbb{Q}[x_1, \dots, x_{81}]$ be the ideal generated by the 891 polynomials $\{G_{i,j}\}_{(i,j)\in E}$ and $\{F_i\}_{i=1,\dots,9}$
- $a = (a_1, \dots, a_{81}) \in V(I)$ iff $a_i \in \{1, \dots, 9\}$ and $a_i \neq a_j$ for $(i,j) \in E$

Sudoku

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- a well posed Sudoku has a unique solution.
- Let $L \subset \{1, \ldots, 81\}$ be the set of pre-assigned places and $\{a_i\}_{i \in L}$ the corresponding numbers of a concrete Sudoku S.
- Then $I_S = I + < \{x_i - a_i\}_{i \in L} >$ is the ideal associated to the Sudoku S.

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- Then $I_S = I + < \{x_i - a_i\}_{i \in L} >$ is the ideal associated to the Sudoku S.
- The reduced Gröbner basis of I_S with respect to the lexicographical ordering has the shape $x_1 - a_1, \ldots, x_{81} - a_{81}$ and $(a_1, \ldots, a_{81})$ is the solution of the Sudoku.

Computeralgebra and finite Groups

Problem: Characterize the class of finite solvable groups G by 2–variable identities.

Computeralgebra and finite Groups

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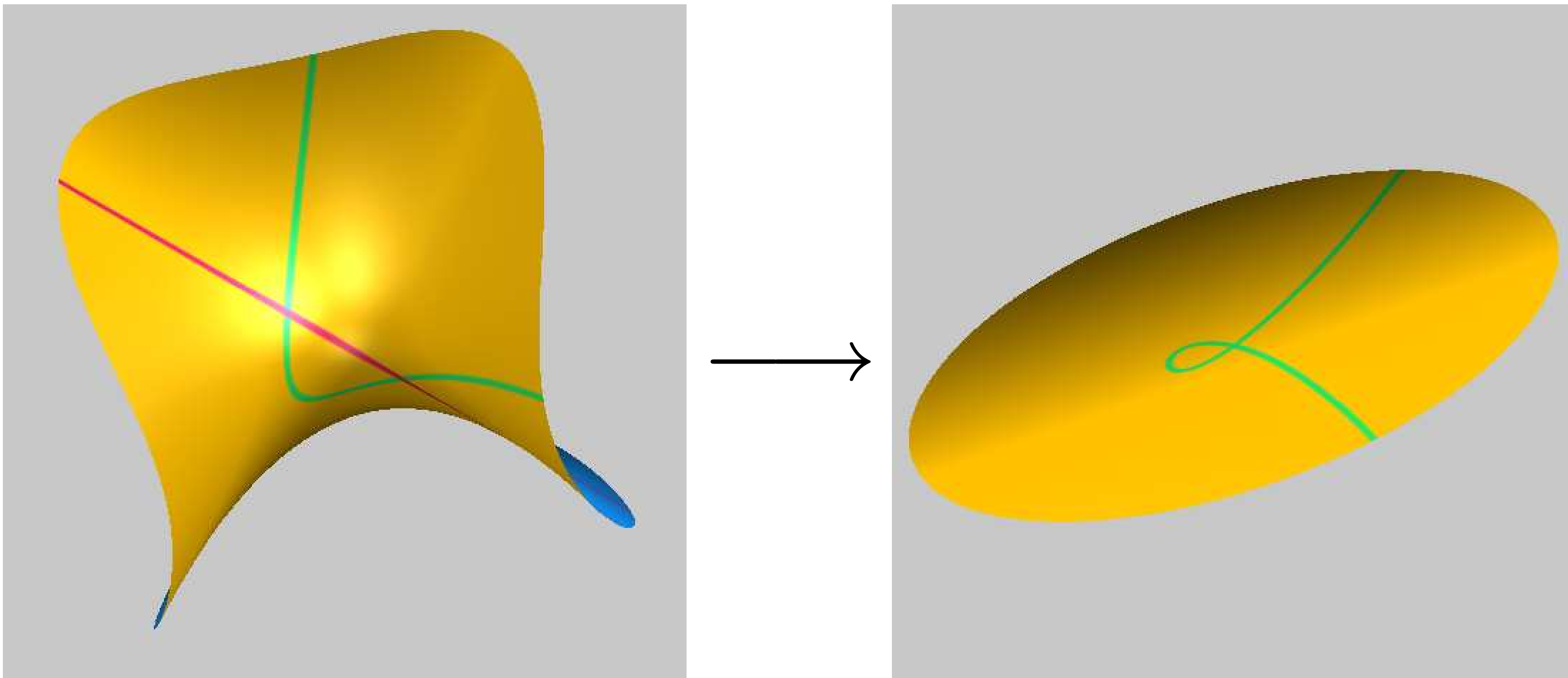
Problem: Characterize the class of finite solvable groups G by 2–variable identities.

Example:

- G is **abelian** $\Leftrightarrow xy = yx \; \forall \; x, y \in G$
- (Zorn, 1930) A finite group G is **nilpotent** $\Leftrightarrow \exists \; n \geq 1$, such that $v_n(x, y) = 1 \; \forall \; x, y \in G$
  (**Engel Identity**)

  $v_1 := [x, y] = xyx^{-1}y^{-1}$ (commutator)
  $v_{n+1} := [v_n, y]$

Blowing up a node



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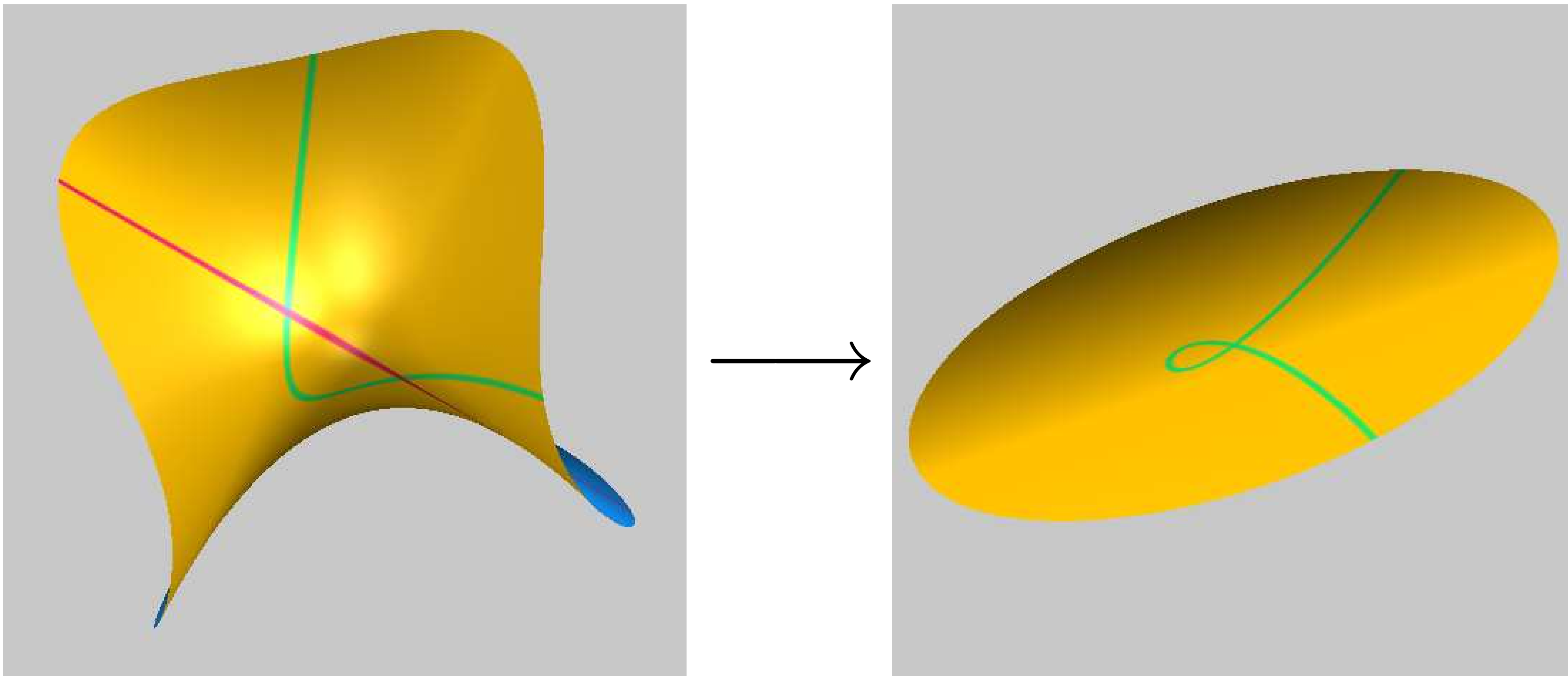
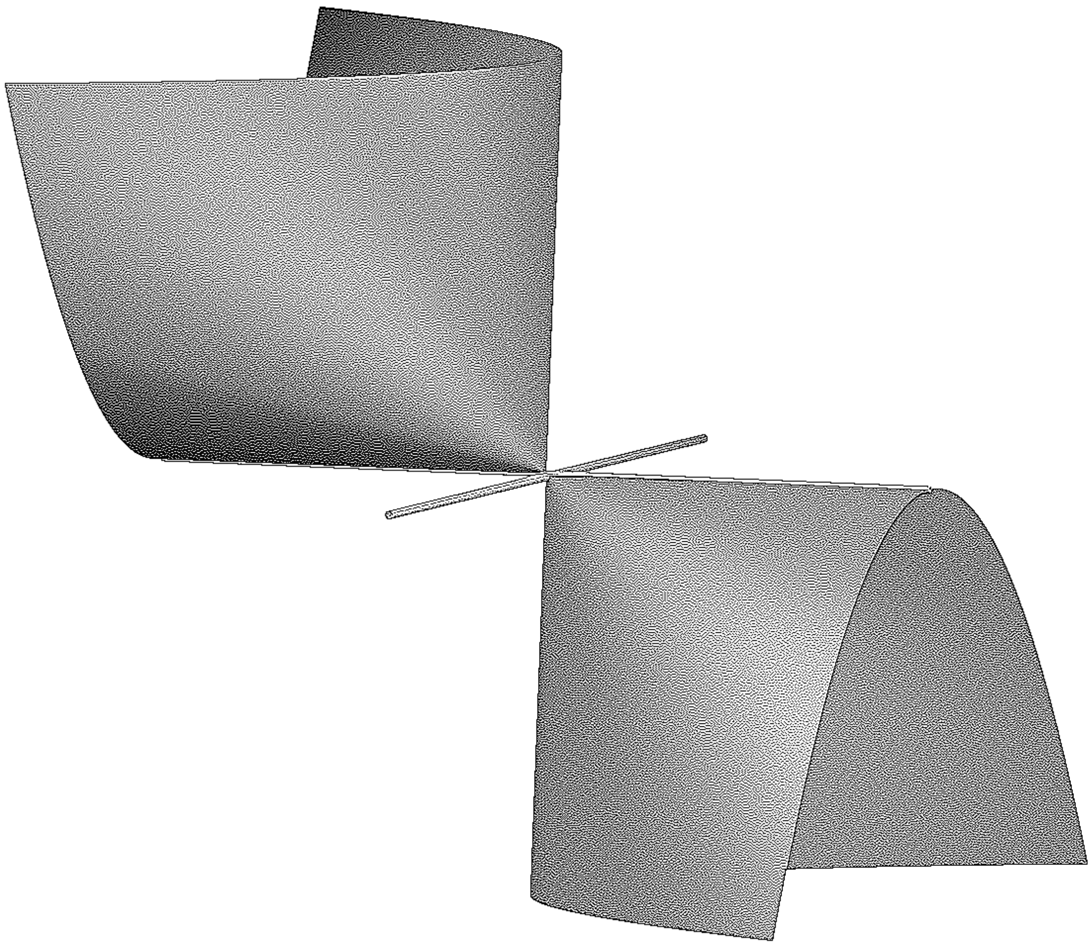


$$\pi : X = \{(x, y; u : v) \in K^2 \times \mathbb{P}^1 \mid xv = yu\} \longrightarrow K^2.$$

Primary decomposition



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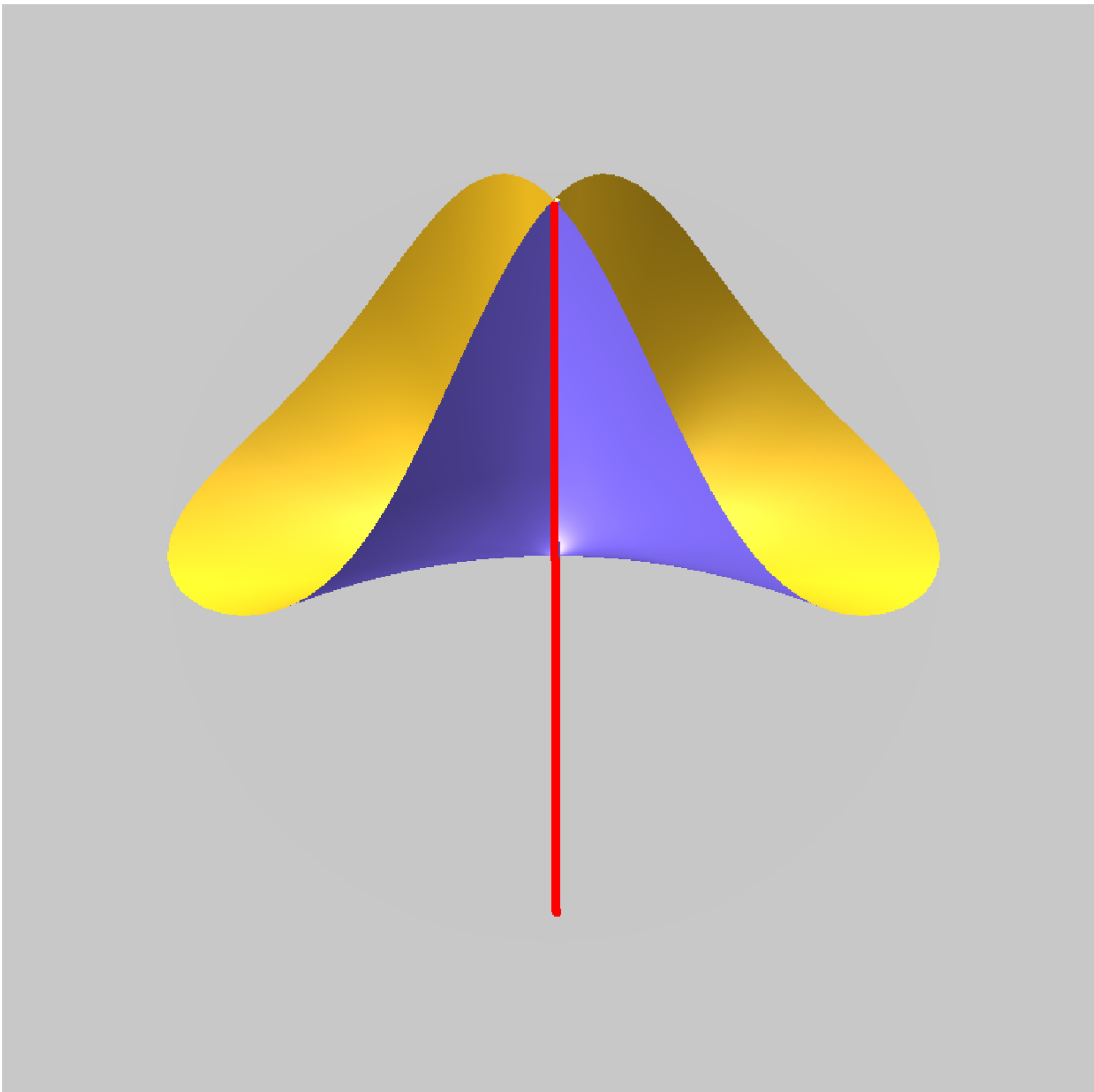
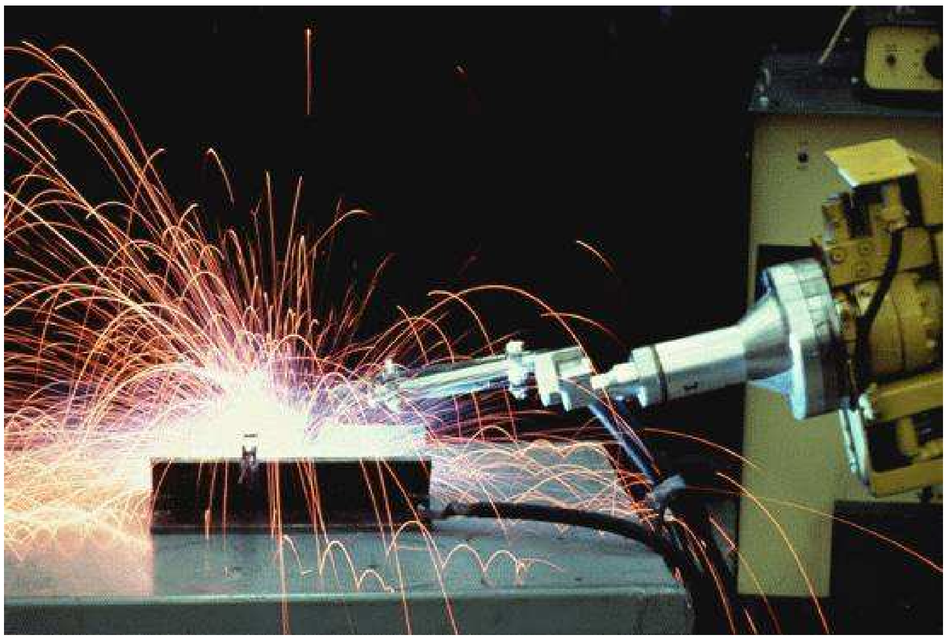


Robotics and the Cycloheptane Molecule



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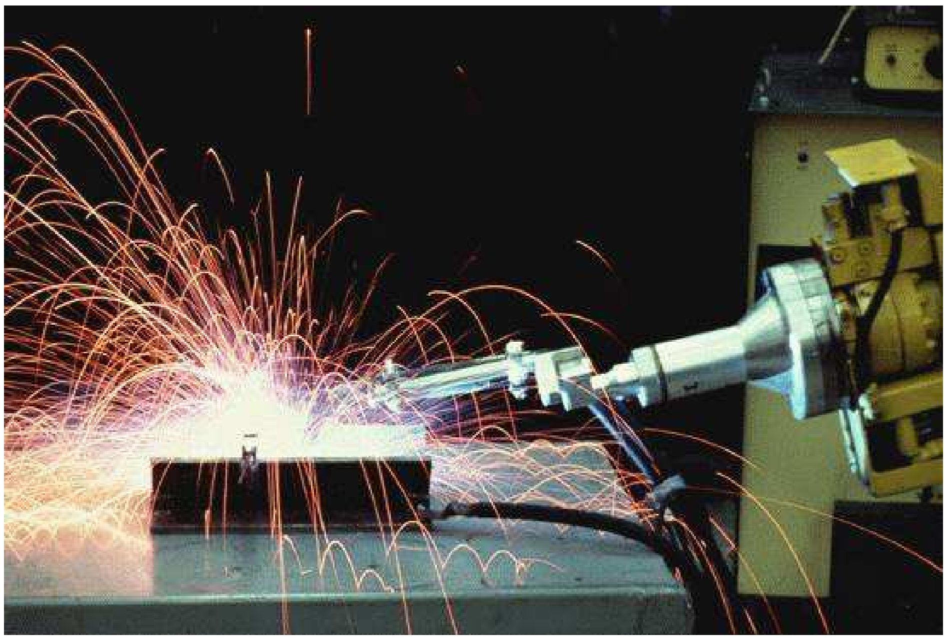

A.H.M. Levelt

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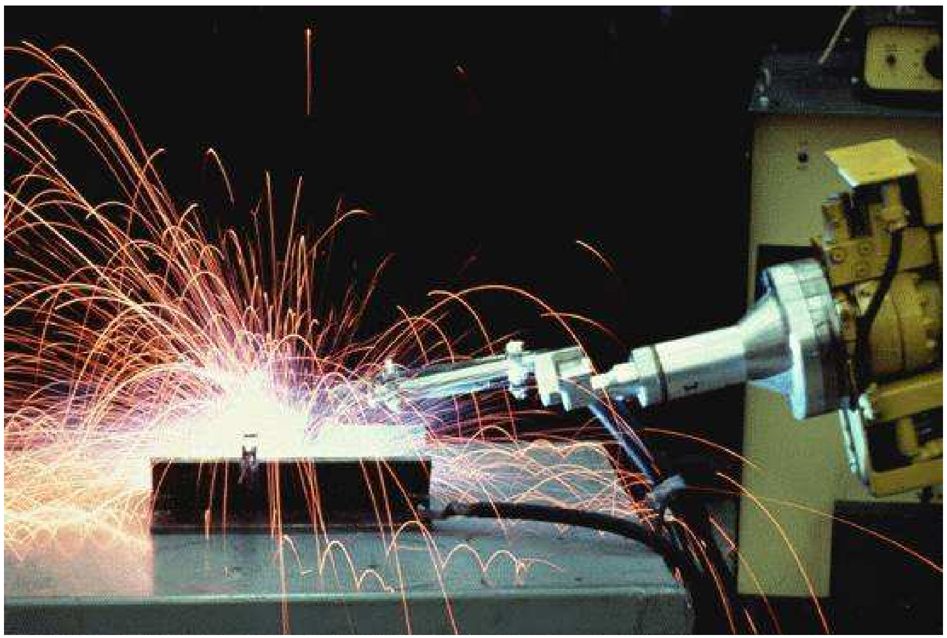

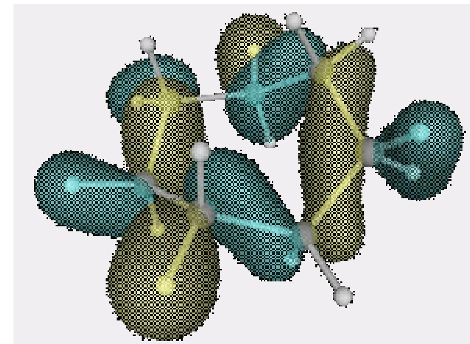

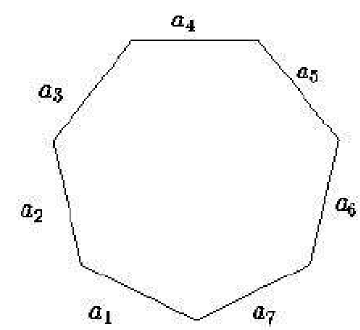

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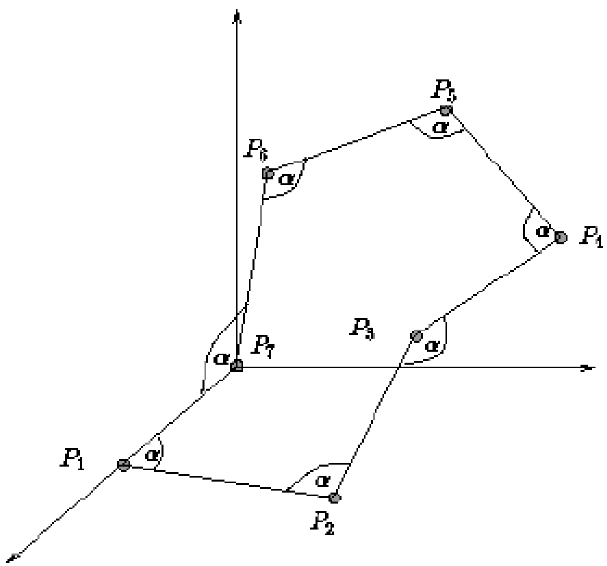
The Heptagon



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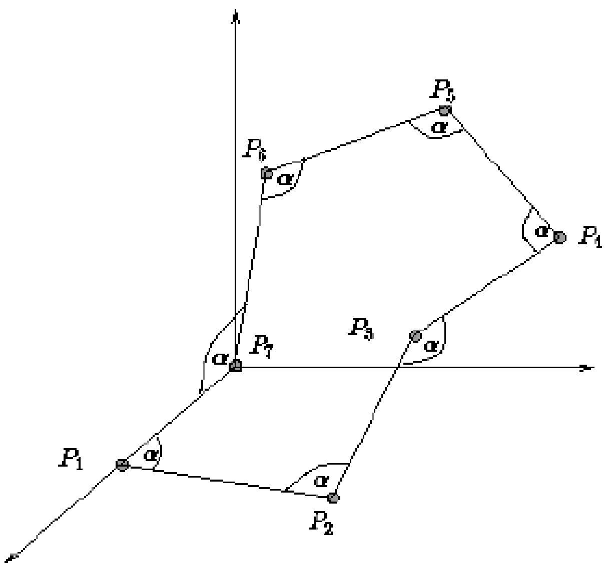

Equations for the vectors:

- $(a_1, a_2) = (a_2, a_3) = \ldots = (a_7, a_1) = c$
- $(a_1, a_1) = (a_2, a_2) = \ldots = (a_7, a_7) = 1$
- $a_1 + a_2 + \ldots + a_7 = 0$

$c = cos(\alpha)$ and $(,)$ is the scalar product.

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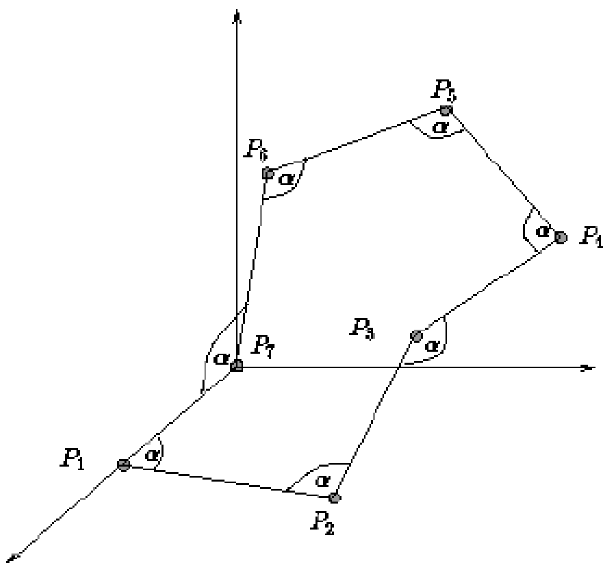

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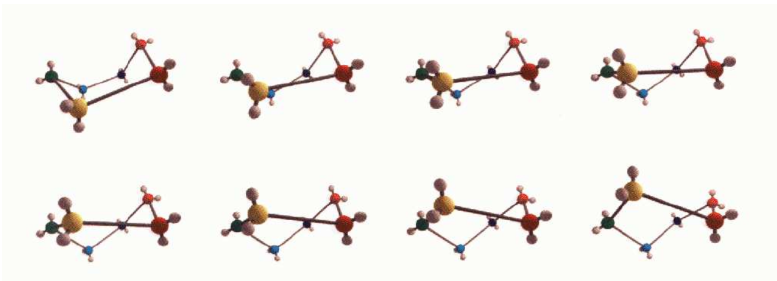
- For $c = 0$: equations for the configuration space of a robot
- For $c = \frac{1}{3}$: equations for the configurations space of a molecule

The Molecule

Equations for the configuration space (in SINGULAR):



```
ring R=0,(v,w,x,y,z),dp;
ideal I=
81y2z2-54wyz+54y2z+54yz2-72w2+198wy-207y2+198wz-225yz-207z2+114w-141y-141z+10,
81w2x2+54w2x+54wx2-54wxz-207w2-225wx-207x2+198wz+198xz-72z2-141w-141x+114z+10,
324vw2x+432vw2+540vwx+432w2x-432wxy-432vwz+324wyz+180vw+846w2-576vx+180wx-
306wy+144xy+144vz-306wz-36yz+12v+585w+12x-318y-318z-79,
81v2w2+54v2w+54vw2-54vwy-207v2-225vw-207w2+198vy+198wy-72y2-141v-141w+114y+10;
```

The Projection

The equations describe a curve in $\mathbb{R}^5$. The projection to the w, x–plane is difficult to compute:

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The equations describe a curve in $\mathbb{R}^5$. The projection to the w, x–plane is difficult to compute:

13343098629642274643741505w20x16+18458805154059402163602552w20x15
+12528539096440613433050772w19x16-307469543636682571308498792w20x14
-308745089273555811810514188w19x15-335770469789305978523636514w18x16

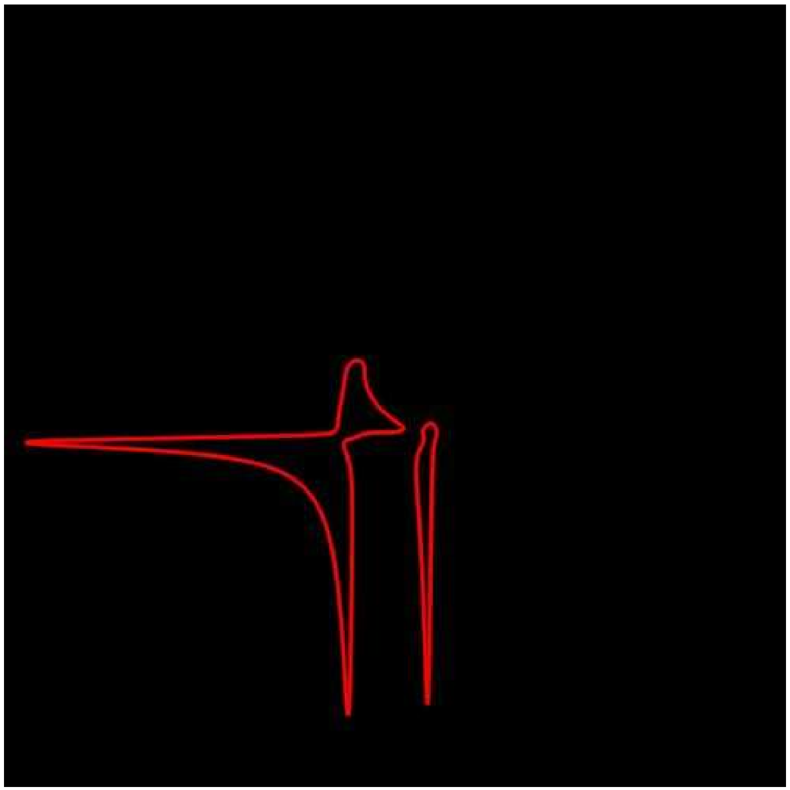
.

.

.

-57603722394732542788396875000w2x-56209703485755917382271875000wx2
-29459059311819369252628125000x3-3456386878638867977468750000w2
-388065077492910629437500000wx-3500955605594366547468750000x2
+1264097844032306972500000000w+1126578705265908772500000000x
+24065849284119685000000000

Projection of the curve to the w, x–plane



```
ideal K = eliminate(I,vyz);
LIB''surf.lib'';
plot(K[1]);
```

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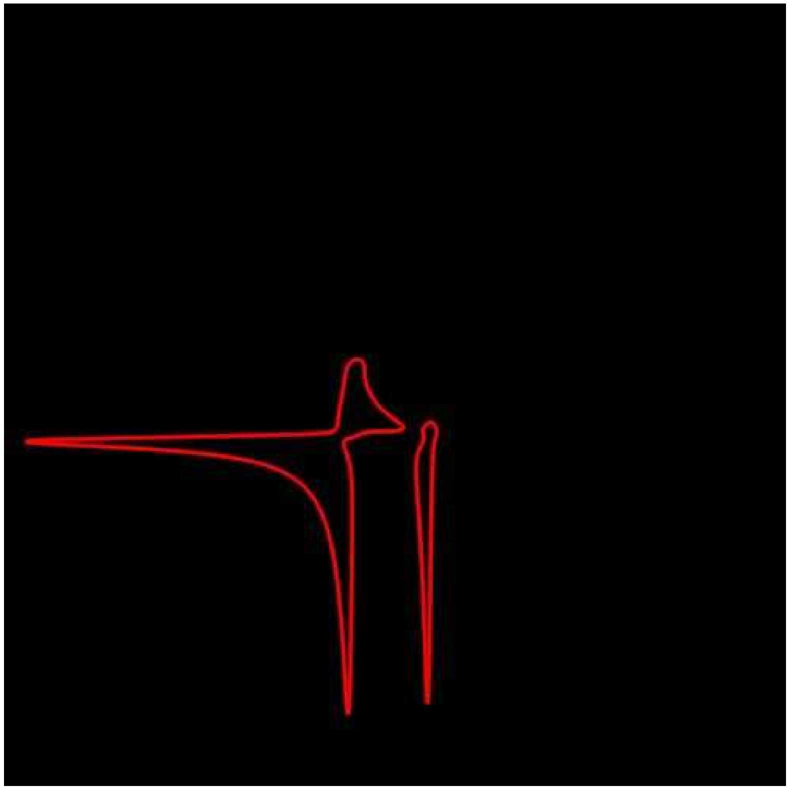

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The curve shows the possible w, x–coordinates of the molecule.

Fields

- rational numbers $\mathbb{Q}$ (charakteristic 0)
- finite fields $\mathbb{Z}/p\mathbb{Z}(p <= 2147483629)$
- finite fields $\mathbb{F}_{p^n}(p^n < 2^{15})$

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- floating point real and complex numbers

Rings

- polynomial rings $K[x_1, \ldots, x_n]$
- localizations $K[x_1, \ldots, x_n]_M$
  M maximal ideal
- factor rings $K[x_1, \ldots, x_n]/J$ oder $K[x_1, \ldots, x_n]_M/J$

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- non-commutative G–algebras
  $K\langle x_1, \ldots, x_n \mid x_j x_i = C_{ij} x_i x_j + D_{ij} \rangle$
  $C_{ij} \in K \;,\; LM(D_{ij}) < x_i x_j$
- factor algebras of G–algebras by two-sided ideals

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- factor algebras of G–algebras by two-sided ideals
- tensor products of the rings above

Algorithms in the Kernel (C/C_{++})

- Standard basis algorithms (Buchberger, SlimGB, factorizing Buchberger, FGLM, Hilbert–driven Buchberger, ...)

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- many libraries: ... control.lib, surf.lib, solve.lib, primdec.lib, resolve.lib,....

Applications

- **mathematical**
  - experimental tool
  - proving theorems

Applications

- **mathematical**
  - experimental tool
  - proving theorems
- **non–mathematical**
  - engineering (glas melting, robotics, chemical models, analog and digital microelectronics)
  - equilibrian problems in economics
  - theoretical physics