# Mathematical Finance

## Introduction to Binary Tree Models, Stochastic Calculus and Black-Scholes Theory

## Solutions to Exercises

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1. The payoff of the put option will be \$0 if the stock price goes up and \$10 if it goes down.

   (a) The value at $T$ of the replicating portfolio $(x, y)$ of cash and stock should be equal to the put payoff

   $$\begin{aligned} 1.05x + 125y &= 0, \\ 1.05x + 95y &= 10. \end{aligned}$$

   The solution is $y = -\frac{1}{3} \approx -0.3333$ and $x \approx 39.6825$.

   (b) The put price at time 0 should be equal to the initial value of the replicating portfolio

   $$x + 100y \approx 6.3492.$$

   (c) The risk neutral probability is

   $$p^* = \frac{105 - 95}{125 - 95} = \frac{1}{3}.$$

   The put price is

   $$\frac{1}{3}\frac{0}{1.05} + \frac{2}{3}\frac{10}{1.05} \approx 6.3492.$$

   (d) If the time 0 put price were \$5, then the portfolio consisting of

   - 3 put options,

- 1 share of stock,
- $-\$115$ in cash,

would be an arbitrage opportunity. It would have initial value \$0 and final value \$4.25 irrespective of whether stock goes up or down.

2. Suppose that

$$C(0) > P(0) + S(0) - \frac{K}{1+r}.$$

In such case, at time 0 we could:

- sell a call option for $C(0)$,
- buy a put option for $P(0)$,
- buy a share of stock for $S(0)$,
- invest the cash amount $\varepsilon = C(0) - P(0) - S(0) > -\frac{K}{1+r}$ at rate $r$.

The value of this portfolio at time 0 would be

$$V(0) = -C(0) + P(0) + S(0) + (C(0) - P(0) - S(0)) = 0.$$

The value of the portfolio at time $T$ would be

$$\begin{aligned} V(T) &= -C(T) + P(T) + S(T) + (1+r)\,(C(0) - P(0) - S(0)) \\ &> -\max(S(T) - K, 0) + \max(K - S(T), 0) + S(T) - K = 0, \end{aligned}$$

which is an arbitrage opportunity. On the other hand, if

$$C(0) < P(0) + S(0) - \frac{K}{1+r},$$

then we could construct the opposite portfolio:

- buy a call option for $C(0)$,
- sell a put option for $P(0)$,
- sell a share of stock for $S(0)$,
- invest the cash amount $-C(0) + P(0) + S(0) > \frac{K}{1+r}$ at rate $r$.

The value of this portfolio at time 0 would be

$$V(0) = C(0) - P(0) - S(0) + (-C(0) + P(0) + S(0)) = 0.$$

The value of the portfolio at time $T$ would be

$$\begin{aligned} V(T) &= C(T) - P(T) - S(T) + (1+r)\,(-C(0) + P(0) + S(0)) \\ &> \max(S(T) - K, 0) - \max(K - S(T), 0) - S(T) + K = 0, \end{aligned}$$

which once again is an arbitrage opportunity. If no arbitrage opportunities can exist, then the only possibility left is that $C(0) = P(0) + S(0) - \frac{K}{1+r}$.

3 See the file: `Exe_3_solution.ods`

4 Let assume that the expiry date of the options is $T > 0$. Using an arbitrage argument, we shall first show that we cannot have

$$C_E(0) > C_A(0). \tag{S.4}$$

If we had (S.4), then our strategy at time $t = 0$ is to hold $v = -1$ of European options, $w = 1$ American options, and invest $x = C_E(0) - C_A(0) > 0$ in cash at rate $r$. The value of our strategy at time zero would be

$$V(0) = vC_E(0) + wC_A(0) + x = 0.$$

At time $T$ the value of our strategy would be

$$\begin{aligned} V(T) &= vC_E(T) + wC_A(T) + (1+r)^N x \\ &= -(S(T) - K)^+ + (S(T) - K)^+ + (1+r)^N (C_E(0) - C_A(0)) \\ &> 0, \end{aligned}$$

which means that this would be an arbitrage opportunity.

Now we will show that we cannot have

$$C_E(0) < C_A(0). \tag{S.5}$$

If we had (S.5) then our strategy at time $t = 0$ is to hold $v = 1$ of European options, $w = -1$ American options, and $x = C_A(0) - C_E(0) > 0$ in cash and $y = 0$ shares of stock. The value of our strategy at time zero would be

$$V(0) = vC_E(0) + wC_A(0) + x = 0.$$

At time step $n\delta \leq N\delta = T$ the holder of the American call option might wish to exercise it. In such case we take a short position in stock and sell it to the holder of the option for $K$ and invest the amount $K$ at rate $r$. We then have the following position

$v' = 1$ of European options
$w' = 0$ of American options (since the holder has exercised his option)
$x' = (1+r)^n (C_A(0) - C_E(0)) + K$ of bonds
$y' = -1$ of stock.

At time $T$ the value of our portfolio will be

$$\begin{aligned} V(T) &= v'C_E(T) + w'C_A(T) + (1+r)^{N-n} x' + y'S(T) \\ &= (S(T) - K)^+ + 0 + (1+r)^N (C_A(0) - C_E(0)) + (1+r)^{N-n} K - S(T) \\ &\geq (S(T) - K)^+ + K - S(T) + (1+r)^N (C_A(0) - C_E(0)) \\ &= (X - S(T))^+ + (1+r)^N (C_A(0) - C_E(0)) \\ &> 0. \end{aligned}$$

On the other hand, if the holder of the American option does not exercise it at all, then we would be holding the original portfolio until time $T$, when its final value would be

$$\begin{aligned} V(T) &= vC_E(T) + wC_A(T) + (1+r)^N x + yS(T) \\ &= C_E(T) - C_A(T) + (1+r)^N (C_A(0) - C_E(0)) \\ &= (1+r)^N (C_A(0) - C_E(0)) \\ &> 0. \end{aligned}$$

This means that we would have have constructed an arbitrage opportunity. The only possibility left is that

$$C_E(0) = C_A(0).$$