Mathematical Finance

Introduction to Binary Tree Models, Stochastic Calculus and Black-Scholes Theory

Solutions to Exercises

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1. The payoff of the put option will be \$0 if the stock price goes up and \$10 if it goes down.

   (a) The value at T of the replicating portfolio (x, y) of cash and stock should be equal to the put payoff

   $$\begin{aligned} 1.05x + 125y &= 0, \\ 1.05x + 95y &= 10. \end{aligned}$$

   The solution is $y = -\frac{1}{3} \approx -0.3333$ and $x \approx 39.6825$.

   (b) The put price at time 0 should be equal to the initial value of the replicating portfolio

   $$x + 100y \approx 6.3492.$$

   (c) The risk neutral probability is

   $$p^* = \frac{105 - 95}{125 - 95} = \frac{1}{3}.$$

   The put price is

   $$\frac{1}{3}\frac{0}{1.05} + \frac{2}{3}\frac{10}{1.05} \approx 6.3492.$$

   (d) If the time 0 put price were \$5, then the portfolio consisting of

   - 3 put options,

- 1 share of stock,
- −\$115 in cash,

would be an arbitrage opportunity. It would have initial value \$0 and final value \$4.25 irrespective of whether stock goes up or down.

2. Suppose that

$$C(0) > P(0) + S(0) - \frac{K}{1+r}.$$

In such case, at time 0 we could:

- sell a call option for $C(0)$,
- buy a put option for $P(0)$,
- buy a share of stock for $S(0)$,
- invest the cash amount $\varepsilon = C(0) - P(0) - S(0) > -\frac{K}{1+r}$ at rate r.

The value of this portfolio at time 0 would be

$$V(0) = -C(0) + P(0) + S(0) + (C(0) - P(0) - S(0)) = 0.$$

The value of the portfolio at time T would be

$$\begin{aligned} V(T) &= -C(T) + P(T) + S(T) + (1+r)\,(C(0) - P(0) - S(0)) \\ &> -\max(S(T) - K, 0) + \max(K - S(T), 0) + S(T) - K = 0, \end{aligned}$$

which is an arbitrage opportunity. On the other hand, if

$$C(0) < P(0) + S(0) - \frac{K}{1+r},$$

then we could construct the opposite portfolio:

- buy a call option for $C(0)$,
- sell a put option for $P(0)$,
- sell a share of stock for $S(0)$,
- invest the cash amount $-C(0) + P(0) + S(0) > \frac{K}{1+r}$ at rate r.

The value of this portfolio at time 0 would be

$$V(0) = C(0) - P(0) - S(0) + (-C(0) + P(0) + S(0)) = 0.$$

The value of the portfolio at time T would be

$$\begin{aligned} V(T) &= C(T) - P(T) - S(T) + (1+r)\,(-C(0) + P(0) + S(0)) \\ &> \max(S(T) - K, 0) - \max(K - S(T), 0) - S(T) + K = 0, \end{aligned}$$

which once again is an arbitrage opportunity. If no arbitrage opportunities can exist, then the only possibility left is that $C(0) = P(0) + S(0) - \frac{K}{1+r}$.