# Mathematical Finance

## Introduction to Binary Tree Models, Stochastic Calculus and Black-Scholes Theory

## Solutions to Exercises

Tomasz Zastawniak
Department of Mathematics
University of York
`tz506@york.ac.uk`

AIMS
4–22 January 2010

1. The payoff of the put option will be \$0 if the stock price goes up and \$10 if it goes down.

   (a) The value at $T$ of the replicating portfolio $(x, y)$ of cash and stock should be equal to the put payoff

   $$\begin{aligned} 1.05x + 125y &= \phantom{1}0, \\ 1.05x + \phantom{1}95y &= 10. \end{aligned}$$

   The solution is $y = -\frac{1}{3} \approx -0.3333$ and $x \approx 39.6825$.

   (b) The put price at time 0 should be equal to the initial value of the replicating portfolio

   $$x + 100y \approx 6.3492.$$

   (c) The risk neutral probability is

   $$p^* = \frac{105 - 95}{125 - 95} = \frac{1}{3}.$$

   The put price is

   $$\frac{1}{3}\frac{0}{1.05} + \frac{2}{3}\frac{10}{1.05} \approx 6.3492.$$

   (d) If the time 0 put price were \$5, then the portfolio consisting of

   - 3 put options,

- 1 share of stock,
- $-\$115$ in cash,

would be an arbitrage opportunity. It would have initial value \$0 and final value \$4.25 irrespective of whether stock goes up or down.

2. Suppose that
$$C(0) > P(0) + S(0) - \frac{K}{1+r}.$$
In such case, at time 0 we could:

   - sell a call option for $C(0)$,
   - buy a put option for $P(0)$,
   - buy a share of stock for $S(0)$,
   - invest the cash amount $\varepsilon = C(0) - P(0) - S(0) > -\frac{K}{1+r}$ at rate $r$.

   The value of this portfolio at time 0 would be
$$V(0) = -C(0) + P(0) + S(0) + (C(0) - P(0) - S(0)) = 0.$$
   The value of the portfolio at time $T$ would be
$$\begin{aligned} V(T) &= -C(T) + P(T) + S(T) + (1+r)\,(C(0) - P(0) - S(0)) \\ &> -\max(S(T) - K, 0) + \max(K - S(T), 0) + S(T) - K = 0, \end{aligned}$$
   which is an arbitrage opportunity. On the other hand, if
$$C(0) < P(0) + S(0) - \frac{K}{1+r},$$
   then we could construct the opposite portfolio:

   - buy a call option for $C(0)$,
   - sell a put option for $P(0)$,
   - sell a share of stock for $S(0)$,
   - invest the cash amount $-C(0) + P(0) + S(0) > \frac{K}{1+r}$ at rate $r$.

   The value of this portfolio at time 0 would be
$$V(0) = C(0) - P(0) - S(0) + (-C(0) + P(0) + S(0)) = 0.$$
   The value of the portfolio at time $T$ would be
$$\begin{aligned} V(T) &= C(T) - P(T) - S(T) + (1+r)\,(-C(0) + P(0) + S(0)) \\ &> \max(S(T) - K, 0) - \max(K - S(T), 0) - S(T) + K = 0, \end{aligned}$$
   which once again is an arbitrage opportunity. If no arbitrage opportunities can exist, then the only possibility left is that $C(0) = P(0) + S(0) - \frac{K}{1+r}$.

3. See the file: `Exe_3_solution.ods`

4. Let assume that the expiry date of the options is $T > 0$. Using an arbitrage argument, we shall first show that we cannot have

$$C_E(0) > C_A(0). \tag{S.10}$$

If we had (S.10), then our strategy at time $t = 0$ is to hold $v = -1$ of European options, $w = 1$ American options, and invest $x = C_E(0) - C_A(0) > 0$ in cash at rate $r$. The value of our strategy at time zero would be

$$V(0) = vC_E(0) + wC_A(0) + x = 0.$$

At time $T$ the value of our strategy would be

$$\begin{aligned} V(T) &= vC_E(T) + wC_A(T) + (1+r)^N x \\ &= -(S(T) - K)^+ + (S(T) - K)^+ + (1+r)^N (C_E(0) - C_A(0)) \\ &> 0, \end{aligned}$$

which means that this would be an arbitrage opportunity.

Now we will show that we cannot have

$$C_E(0) < C_A(0). \tag{S.11}$$

If we had (S.11) then our strategy at time $t = 0$ is to hold $v = 1$ of European options, $w = -1$ American options, and $x = C_A(0) - C_E(0) > 0$ in cash and $y = 0$ shares of stock. The value of our strategy at time zero would be

$$V(0) = vC_E(0) + wC_A(0) + x = 0.$$

At time step $n\delta \leq N\delta = T$ the holder of the American call option might wish to exercise it. In such case we take a short position in stock and sell it to the holder of the option for $K$ and invest the amount $K$ at rate $r$. We then have the following position

$v' = 1$ of European options
$w' = 0$ of American options (since the holder has exercised his option)
$x' = (1+r)^n (C_A(0) - C_E(0)) + K$ of bonds
$y' = -1$ of stock.

At time $T$ the value of our portfolio will be

$$\begin{aligned} V(T) &= v'C_E(T) + w'C_A(T) + (1+r)^{N-n} x' + y'S(T) \\ &= (S(T) - K)^+ + 0 + (1+r)^N (C_A(0) - C_E(0)) + (1+r)^{N-n} K - S(T) \\ &\geq (S(T) - K)^+ + K - S(T) + (1+r)^N (C_A(0) - C_E(0)) \\ &= (X - S(T))^+ + (1+r)^N (C_A(0) - C_E(0)) \\ &> 0. \end{aligned}$$

On the other hand, if the holder of the American option does not exercise it at all, then we would be holding the original portfolio until time $T$, when its final value would be

$$\begin{aligned} V(T) &= vC_E(T) + wC_A(T) + (1+r)^N x + yS(T) \\ &= C_E(T) - C_A(T) + (1+r)^N (C_A(0) - C_E(0)) \\ &= (1+r)^N (C_A(0) - C_E(0)) \\ &> 0. \end{aligned}$$

This means that we would have constructed an arbitrage opportunity. The only possibility left is that

$$C_E(0) = C_A(0).$$

5. We can assume without loss of generality that $0 \leq s \leq t$. Then

$$\begin{aligned} \mathbb{E}\left[B(s)B(t)\right] &= \mathbb{E}\left[B(s)(B(t) - B(s))\right] + \mathbb{E}\left[B(s)^2\right] \\ &= \mathbb{E}\left[B(s)\right]\mathbb{E}\left[B(t) - B(s)\right] + \mathbb{E}\left[B(s)^2\right] \\ &= 0 \times 0 + s = s = \min(s,t). \end{aligned}$$

6. We need to verify that $W(t)$ satisfies the four conditions in the definition of Brownian motion.

   - $W(0) = \frac{1}{c}B(c^2 0) = 0$.
   - If $B(t)$ is continuous as a function of $t$, then so is $W(t) = \frac{1}{c}B(c^2 t)$.
   - The independence of the increments $W(u) - W(v) = \frac{1}{c}\left(B(c^2 u) - B(c^2 v)\right)$ and $W(t) - W(s) = \frac{1}{c}\left(B(c^2 t) - B(c^2 s)\right)$ for $0 \leq v \leq w \leq s \leq t$ follows from the independence of $B(c^2 u) - B(c^2 v)$ and $B(c^2 t) - B(c^2 s)$.
   - For any $0 \leq s < t$

     $$W(t) - W(s) = \frac{1}{c}\left[B(c^2 t) - B(c^2 s)\right].$$

     Since $B(c^2 t) - B(c^2 s) \sim N(0, c^2 t - c^2 s)$ we find immediately that $W(t) - W(s) \sim N(0, t - s)$.

7. We compute the second moment

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{N-1}\left(B(t_{i+1})-B(t_i)\right)^2-T\right)^2\right]\\
&=\mathbb{E}\left[\left(\sum_{i=1}^{N-1}\Big[\left(B(t_{i+1})-B(t_i)\right)^2-(t_{i+1}-t_i)\Big]\right)^2\right]\\
&=\sum_{i=1}^{N-1}\mathbb{E}\left[\Big(\left(B(t_{i+1})-B(t_i)\right)^2-(t_{i+1}-t_i)\Big)^2\right]\\
&\quad\text{the variance of a sum of independent random variables is the sum of variances}\\
&=\sum_{i=1}^{N-1}\mathbb{E}\Big[\left(B(t_{i+1})-B(t_i)\right)^4-2\left(B(t_{i+1})-B(t_i)\right)^2(t_{i+1}-t_i)+(t_{i+1}-t_i)^2\Big]\\
&=\sum_{i=1}^{N-1}\Big(3\left(t_{i+1}-t_i\right)^2-2\left(t_{i+1}-t_i\right)^2+\left(t_{i+1}-t_i\right)^2\Big)\\
&\quad\text{computing the 4th and 2nd moments of } B(t_{i+1})-B(t_i)\\
&=2\sum_{i=1}^{N-1}\left(t_{i+1}-t_i\right)^2=2\sum_{i=1}^{N-1}\left(\frac{T}{N}\right)^2=2N\left(\frac{T}{N}\right)^2\to 0\quad\text{as } N\to\infty.
\end{aligned}
$$

This shows that

$$\sum_{i=1}^{N-1}\left(B(t_{i+1})-B(t_i)\right)^2\to T\quad\text{as } N\to\infty$$

in $L^2$.

8. Take $f(t,x)=Ce^{ax-\frac{1}{2}a^2t}$. Then $X(t)=f(t,B(t))$ and

$$f'_t(t,x)=-\frac{1}{2}a^2f(t,x),\quad f'_x(t,x)=af(t,x),\quad f''_{xx}(t,x)=a^2f(t,x).$$

Substituting these into the Itô formula

$$df(t,B(t))=\left(f'_t(t,B(t))+\frac{1}{2}f''_{xx}(t,B(t))\right)dt+f'_x(t,B(t))dB(t),$$

we get

$$dX(t)=\left(-\frac{1}{2}a^2X(t)+\frac{1}{2}a^2X(t)\right)dt+aX(t)dB(t)=aX(t)dB(t).$$

9. For $f(s,B(s))=u(t-s,x+B(s))$ the Itô formula

$$
\begin{aligned}
f(t,B(t))=f(0,B(0))+\int_0^t\left(f'_s(s,B(s))+\frac{1}{2}f''_{xx}(s,B(s))\right)ds&\\
+\int_0^t f'_x(s,B(s))dB(s)&
\end{aligned}
$$

gives

$$\begin{aligned} u(0,x+B(t)) &= u(t,x) + \int_0^t \left(-u_t'(t-s,x+B(s)) + \frac{1}{2}u_{xx}''(t-s,x+B(s))\right)ds \\ &\qquad + \int_0^t u_x'(t-s,x+B(s))dB(s) \\ &= u(t,x) + \int_0^t u_x'(t-s,x+B(s))dB(s) \end{aligned}$$

so that

$$\mathbb{E}\left[u(0,x+B(t))\right] = u(t,x).$$

Since $u(0,x+B(t)) = \varphi(x+B(t))$, we obtain

$$u(t,x) = \mathbb{E}\left[\varphi(x+B(t))\right].$$

10. Let $X(t) = \int_0^t sdB(s) - \frac{1}{6}t^3$, that is, $dX(t) = tdB(t) - \frac{1}{2}t^2dt$, and let $f(t,x) = e^x$. Then

$$e^{\int_0^t sdB(s) - \frac{1}{6}t^3} = f(t,X(t))$$

so that

$$\begin{aligned} de^{X(t)} &= df(t,X(t)) \\ &= \left(f_t'(t,X(t)) + \frac{1}{2}t^2 f_{xx}''(t,X(t))\right)dt + f_x'(t,X(t))dX(t) \\ &= \frac{1}{2}t^2e^{X(t)}dt + e^{X(t)}\left(tdB(t) - \frac{1}{2}t^2dt\right) \\ &= e^{X(t)}tdB(t) \end{aligned}$$

It follows that

$$\mathbb{E}\left[e^{X(T)}\right] = e^{X(0)},$$

which means that

$$\mathbb{E}\left[e^{\int_0^T tdB(t) - \frac{1}{6}T^3}\right] = 1.$$

As a result,

$$\mathbb{E}\left[e^{\int_0^T tdB(t)}\right] = e^{\frac{1}{6}T^3}.$$