3.8 Black-Scholes Equation

Let us take a sufficiently regular function $u(t,x)$. Since

$$\begin{aligned} dS(t) &= \left(\mu + \frac{1}{2}\sigma^2\right) S(t)dt + \sigma S(t)dB(t) \\ &= rS(t)dt + \sigma S(t)dB^*(t), \end{aligned}$$

the Itô formula gives

$$\begin{aligned} & d\left[e^{-rt}u(t,S(t))\right] \\ &= e^{-rt}\left[-ru(t,S(t)) + u_t'(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u_{xx}''(t,S(t))\right] dt + u_x'(t,S(t)))dS(t) \\ &= e^{-rt}\left[-ru(t,S(t)) + u_t'(t,S(t)) + \left(\mu + \frac{1}{2}\sigma^2\right) S(t)u_x'(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u_{xx}''(t,S(t))\right] dt \\ &\quad + \sigma e^{-rt}S(t)u_x'(t,S(t)))dB(t) \\ &= e^{-rt}\left[-ru(t,S(t)) + u_t'(t,S(t)) + rS(t)u_x'(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u_{xx}''(t,S(t))\right] dt \\ &\quad + \sigma e^{-rt}S(t)u_x'(t,S(t)))dB^*(t). \end{aligned}$$

If $u(t,x)$ satisfies the equation

$$u_t + rxu_x' + \frac{1}{2}\sigma^2 x^2 u_{xx}'' = ru, \tag{3.12}$$

then

$$d\left[e^{-rt}u(t,S(t))\right] = \sigma e^{-rt}S(t)u_x'(t,S(t)))dB^*(t),$$

that is,

$$e^{-rT}u(T,S(T)) = u(0,S(0)) + \int_0^T \sigma e^{-rt}S(t)u_x'(t,S(t)))dB^*(t).$$

Applying expectation under P^* to both sides of this equality gives

$$\mathbb{E}^*\left[e^{-rT}u(T,S(T))\right] = u(0,S(0)).$$

If, in addition, $u(t,x)$ satisfies the final condition

$$u(T,x) = h(x),$$

then

$$u(0,S(0)) = \mathbb{E}^*\left[e^{-rT}u(T,S(T))\right] = \mathbb{E}^*\left[e^{-rT}h(S(T))\right].$$

The partial differential equation (3.12) is the famous Black-Scholes equation. We have established the following result.

Theorem 48 *If a European type derivative security with exercise time T and payoff $h(S(T))$ can be replicated by a self financing strategy, and if $u(t,x)$ is a solution to the final value problem*

$$
\begin{aligned}
&u_t + rxu'_x + \frac{1}{2}\sigma^2 x^2 u''_{xx} = ru, \\
&u(T,x) = h(x),
\end{aligned}
$$

for the Black-Scholes equation, then the value of the derivative security at time 0 *is given by*

$$D(0) = u(0, S(0)).$$

3.9 Replicating Strategy

Let $u(t,x)$ be a solution to the final value problem

$$
\begin{aligned}
&u_t + rxu'_x + \frac{1}{2}\sigma^2 x^2 u''_{xx} = ru, \\
&u(T,x) = h(x),
\end{aligned}
$$

for the Black-Scholes equation. It is not very hard to show that such a solution exists by verifying that it is given by the explicit formula

$$u(t,x) = e^{r(T-t)}\mathbb{E}\left[xe^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma X}\right],$$

where X is a random variable with normal distribution $N(0, T-t)$.

Next, apply the Itô formula to get

$$du(t,S(t)) = \left[u'_t(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u''_{xx}(t,S(t))\right]dt + u'_x(t,S(t)))dS(t),$$

and define

$$
\begin{aligned}
x(t) &:= \frac{1}{r}\left[u'_t(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u''_{xx}(t,S(t))\right], \\
y(t) &:= u'_x(t,S(t))),
\end{aligned}
$$

and

$$V(t) := x(t) + S(t)y(t).$$

These are clearly adapted processes. Since $u(t,x)$ satisfies the Black-Scholes equation, we have

$$
\begin{aligned}
V(t) &= x(t) + S(t)y(t) \\
&= \frac{1}{r}\left[u'_t(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u''_{xx}(t,S(t))\right] + S(t)u'_x(t,S(t))) \\
&= [u(t,S(t)) - S(t)u'_x(t,S(t))] + S(t)u'_x(t,S(t))) \\
&= u(t,S(t)).
\end{aligned}
$$

It follows that

$$\begin{aligned}dV(t) &= du(t,S(t))\\ &= \left[u'_t(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 u''_{xx}(t,S(t))\right]dt + u'_x(t,S(t)))dS(t)\\ &= rx(t)dt + y(t)dS(t).\end{aligned}$$

We have proved that $(x(t), y(t))$ is a self financing strategy. Moreover, from the final condition $u(T,x) = h(x)$,

$$V(T) = u(T,S(T)) = h(S(T)),$$

so that $(x(t), y(t))$ is in fact a replicating strategy for the European option with payoff $h(S(T))$.

We have not only demonstrated that a replicating strategy exists, but were able to obtain formulae for the strategy in terms of the solution $u(t,x)$ to the final value problem for the Black-Scholes equation.

Example 49 We shall compute the replicating strategy at time 0 for a European call option with strike price K and exercise time T. From the Black-Scholes formula

$$C(0) = S(0)N(d_1) - e^{-rT}KN(d_2)$$

with

$$d_1 = \frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

we can see that

$$u(t,x) = xN\left(\frac{\ln\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - e^{-rT}KN\left(\frac{\ln\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

Differentiating with respect to x we get

$$y(0) = u'_x(0,x) = \cdots = N\left(\frac{\ln\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

We have omitted here a lengthy but straightforward computation of the derivative (an elementary exercise in using the chain rule). As a result, we obtain

$$\begin{aligned}y(0) &= u'_x(0,S(0))\\ &= N(d_1),\\ x(0) &= C(0) - S(0)y(0)\\ &= e^{-rT}KN(d_2).\end{aligned}$$

4 Monte Carlo Simulation

Consider an option with discounted payoff H, a random variable on Ω with finite risk neutral expectation $\mu = \mathbb{E}^*(H)$, which is the option price, and finite variance $0 < \sigma^2 = \mathrm{Var}^*(H)$ under the risk neutral probability P^*.

4.1 Sample Mean

Let $\omega_1, \ldots, \omega_N \in \Omega$ be a sequence of independent scenarios sampled from Ω under the risk neutral probability, so that $H_1 = H(\omega_1), \ldots, H_N = H(\omega_N)$ are independent random variables with the same distribution as H under P^*. Then the sample mean

$$\mu_N = \frac{1}{N} \sum_{n=1}^{N} H_n$$

is a random variable, whose expectation is equal to,

$$\mathbb{E}^*(\mu_N) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}^*(H_n) = \mu.$$

As a result, μ_N is an unbiased estimator for μ. Moreover, by the strong law of large numbers, $\mu_N \to \mu$ almost surely as $N \to \infty$.

In a nutshell, the Monte Carlo method involves simulating sufficiently many independent scenarios $\omega_1, \ldots, \omega_N$ under the risk neutral probability, computing the sample mean μ_N for these scenarios, and using it as an approximation for the option price μ.

4.2 Standard Error of Monte Carlo Simulation

This raises the question of accuracy of such an approximation. Accuracy can be quantified as the standard deviation σ_N of the sample mean μ_N. It is related to the standard deviation σ of H by

$$\sigma_N = \frac{\sigma}{\sqrt{N}}.$$

Indeed,

$$\sigma_N^2 = \mathrm{Var}^*(\mu_N) = \frac{1}{N^2} \sum_{n=1}^{N} \mathrm{Var}^*(H_n) = \frac{\sigma^2}{N}$$

by the independence of $H_1, \ldots, H_N$.

The accuracy $\sigma_N = \frac{\sigma}{\sqrt{N}}$ of Monte Carlo simulation is inversely proportional to the square root of sample size N. To improve the accuracy by one decimal point, the sample size would need to increase 100 times.

In practice, for more complicated payoffs, we may not know σ. It is typically harder to compute σ than the option price μ. However, Monte Carlo simulation can be used once again, this time to approximate the variance σ^2. To this end we need the unbiased estimator

$$S_N^2 = \frac{1}{N-1}\sum_{n=1}^{N}(H_n - \mu_N)^2$$

for σ^2. The quantity $\frac{S_N}{\sqrt{N}}$, known as the standard error of Monte Carlo simulation, can then be used as an approximation for the accuracy $\sigma_N = \frac{\sigma}{\sqrt{N}}$ of Monte Carlo simulation.

That S_N^2 is indeed an unbiased estimator for σ^2 can be checked as follows:

$$\begin{aligned}
\mathbb{E}^*\left(S_N^2\right) &= \frac{1}{N-1}\mathbb{E}^*\sum_{n=1}^{N}(H_n - \mu_N)^2 \\
&= \frac{1}{N-1}\mathbb{E}^*\sum_{n=1}^{N}((H_n - \mu) - (\mu_N - \mu))^2 \\
&= \frac{1}{N-1}\mathbb{E}^*\sum_{n=1}^{N}\left((H_n - \mu)^2 - 2(H_n - \mu)(\mu_N - \mu) + (\mu_N - \mu)^2\right) \\
&= \frac{1}{N-1}\mathbb{E}^*\left(\sum_{n=1}^{N}(H_n - \mu)^2 - 2N(\mu_N - \mu)^2 + N(\mu_N - \mu)^2\right) \\
&= \frac{1}{N-1}\sum_{n=1}^{N}\mathbb{E}^*(H_n - \mu)^2 - \frac{N}{N-1}\mathbb{E}^*(\mu_N - \mu)^2 \\
&= \frac{N}{N-1}\sigma^2 - \frac{N}{N-1}\sigma_N^2 \\
&= \sigma^2.
\end{aligned}$$

4.3 Confidence Interval

For any confidence level c with $0 < c < 1$ the percentile $z_{1-\frac{c}{2}}$ is defined as the unique number for which $P(-z_{1-\frac{c}{2}} \leq Z \leq z_{1-\frac{c}{2}}) = 1 - c$ for any random variable Z with standard normal distribution $N(0,1)$.

According to the Central Limit Theorem, $\frac{\mu_N-\mu}{\sigma/\sqrt{N}} = \frac{\mu_N-\mu}{\sigma_N}$ converges in distribution to a random variable with standard normal distribution $N(0,1)$ as $N \to \infty$. In effect, as $N \to \infty$

$$P\left(\mu_N - \frac{\sigma}{\sqrt{N}}z_{1-\frac{c}{2}} \leq \mu \leq \mu_N + \frac{\sigma}{\sqrt{N}}z_{1-\frac{c}{2}}\right) = P\left(-z_{1-\frac{c}{2}} \leq \frac{\mu_N - \mu}{\sigma/\sqrt{N}} \leq z_{1-\frac{c}{2}}\right) \to 1 - c.$$

We say that $\left[\mu_N - \frac{\sigma}{\sqrt{N}}z_{1-\frac{c}{2}}, \mu_N + \frac{\sigma}{\sqrt{N}}z_{1-\frac{c}{2}}\right]$ is a $100(1-c)\%$ confidence interval for μ. For large N, the option price μ will lie within this interval with probability of about $1 - c$.

In practice, standard error of Monte Carlo simulation $\frac{S_N}{\sqrt{N}}$ is used in place of $\frac{\sigma}{\sqrt{N}}$ to approximate the end-points of the confidence interval.

4.4 Path Independent Options

We shall use Monte Carlo simulation to price a path independent European option with expiry time T and arbitrary payoff $h(S(T))$ in the Black-Scholes model with the underlying stock price given by

$$S(t) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B^*(t)}$$

for $t \geq 0$, with constant interest rate $r > 0$ and initial stock price $S(0)$, where $B^*(t)$ is Brownian motion under the probability P^*.

The option price can be expressed as

$$\mu = \mathbb{E}^*(e^{-rT}h(S(T))) = \mathbb{E}^*\left(e^{-rT}h\left(S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma B^*(T)}\right)\right)$$

with expectation computed under the risk neutral probability.

Because $B^*(T)$ has the normal distribution $N(0,T)$ with mean 0 and variance T under P^*, Monte Carlo simulation can be implemented by generating a sequence of numbers $x_1, x, \ldots, x_N$ sampled independently from the standard normal distribution $N(0,1)$ and computing

$$\mu_N = \frac{1}{N}e^{-rT}\sum_{n=1}^{N}h\left(S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x_n}\right)$$

as an approximation for the option price.

4.5 Box-Muller Algorithm

To implement Monte Carlo simulation for European options in the Black-Scholes model, we need to generate random samples from the standard normal distribution $N(0,1)$.

Many programming languages, including C++, provide a (pseudo)random number generator to produce random samples from the uniform distribution on the unit interval $[0,1]$. We take this as a starting point.

One of the simpler ways to transform a random variable x with uniform distribution on $[0,1]$ into a random variable X with standard normal distribution $N(0,1)$ is to apply the inverse distribution function

$$X = N^{-1}(x),$$

where $N(x)$ is given by 3.11. However, computationally this is not particularly efficient.

A much more efficient method is provided by the Box-Muller algorithm, which can be formulated in two equivalent ways.

Basic Form

- Generate two independent random numbers x, y, each with uniform distribution on $[0, 1]$;
- Put

$$X = \sqrt{-2\ln x}\cos(2\pi y),$$
$$Y = \sqrt{-2\ln x}\sin(2\pi y).$$

Then X, Y are independent with distribution $N(0, 1)$.

Polar Form

- Generate two independent random numbers u, v, each with uniform distribution on $[0, 1]$;
- Discard these numbers if $u^2 + v^2 > 1$;
- If $u^2 + v^2 \leq 1$, then set

$$R = u^2 + v^2,$$
$$U = u\sqrt{\frac{-2\ln R}{R}},$$
$$V = v\sqrt{\frac{-2\ln R}{R}}.$$

Then U, V are independent with distribution $N(0, 1)$.

4.6 Antithetic Sampling

An obvious way to reduce the standard error of Monte Carlo simulation $\frac{S_N}{\sqrt{N}} \approx \frac{\sigma}{\sqrt{N}}$ and so improve accuracy is to increase the number N of simulated scenarios. This is computationally inefficient since the error decreases as $N^{-1/2}$.

Another way is to improve accuracy would be to reduce σ. This can be accomplished, for example, by simulating a random variable H' with the same expectation μ as the original discounted payoff H but smaller standard deviation $\sigma' < \sigma$.

Whenever the payoff can be expressed as a monotone function $H = f(X)$ of variable X whose distribution is symmetric about 0, as is the case for a call or put option in the Black-Scholes Model, a reduction in σ can be achieved without a significant computational overhead by taking

$$H' = \frac{1}{2}(f(X) + f(-X)).$$

Here X and$-X$ are referred to as antithetic random variables. By symmetry, they share the same probability distribution.

Clearly,

$$\mu = \mathbb{E}(H) = \mathbb{E}(H').$$

To show that $\sigma' < \sigma$ we can argue as follows. Let Y be a random variable with the same distribution as X and independent of X. Since f is a monotone function, $[f(X) - f(Y)]\,[f(-X) - f(-Y)] \le 0$, so

$$\begin{aligned}
0 &\ge \mathbb{E}\,[f(X) - f(Y)]\,[f(-X) - f(-Y)] \\
&= \mathbb{E}\,[f(X)f(-X)] + \mathbb{E}\,[f(Y)f(-Y)] \\
&\qquad\qquad - \mathbb{E}\,[f(X)]\,\mathbb{E}\,[f(-Y)] - \mathbb{E}\,[f(-X)]\,\mathbb{E}\,[f(Y)] \\
&= 2\mathbb{E}\,[f(X)f(-X)] - 2\mathbb{E}\,[f(X)]\,\mathbb{E}\,[f(-X)] \\
&= 2\mathrm{Cov}(f(X), f(-X)).
\end{aligned}$$

It follows that

$$\begin{aligned}
{\sigma'}^2 &= \mathrm{Var}(H') \\
&= \frac{1}{4}\left(\mathrm{Var}f(X) + \mathrm{Var}f(-X) + 2\mathrm{Cov}(f(X), f(-X)\right) \\
&\le \frac{1}{4}\left(\sigma^2 + \sigma^2\right) = \frac{1}{2}\sigma^2 < \sigma^2.
\end{aligned}$$

We have shown that ${\sigma'}^2 \le \frac{1}{2}\sigma^2$. With antithetic variables, we can reduce the sample size N and therefore computing time by a factor or 2 and still obtain the same accuracy as the original Monte Carlo simulation. In practice even better improvement can often be achieved than this theoretical bound.

4.7 Path Dependent Options

Examples of path dependent options include barrier options, for instance an up-and-out call with discounted payoff

$$H = \begin{cases} e^{-rT}\max\{S(T) - K, 0\} & \text{if } S(t) < B \text{ for all } t \in [0, T], \\ 0 & \text{if } S(t) \ge B \text{ for some } t \in [0, T]. \end{cases}$$

The expiry time for this option is T, strike price is K, and B is called the barrier. The option will pay noting if the stock price exceeds the barrier at any time between 0 and T. Otherwise the payoff is that of a European call. The payoff H for a barrier option depends on the underlying asset price path $S(t), t \in [0, T]$ and not just on the final price $S(T)$ as for path independent options.

This requires simulating of the entire path $S(t), t \in [0, T]$. In the Black-Scholes model we have

$$S(t) = S(0)e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B^*(t)}$$

for $t \ge 0$, with constant interest rate $r > 0$ and initial stock price $S(0)$, where $B^*(t)$ is Brownian motion under the probability P^*, so we need to simulate the paths of Brownian motion.

Numerically, this can be accomplished by dividing the time interval $[0, T]$ into M parts of length $\Delta t = \frac{T}{M}$, generating M independent random variables $X_1, \ldots, X_M$, each with the standard normal distribution $N(0,1)$ and using

$$B^*(t) \approx \sqrt{\frac{T}{M}} \sum_{m=1}^{[t/M]} X_m$$

as an approximation for $B^*(t)$ and, correspondingly,

$$S(t) \approx S(0) e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma \sqrt{\frac{T}{M}} \sum_{m=1}^{[t/M]} X_m}$$

as an approximation for $S(t)$. Here $[t/M]$ denotes the integer part of t/M.

4.8 Variance Reduction Techniques

Monte Carlo simulation for path dependent options can be computationally intensive. To reduce the error $\frac{\sigma}{\sqrt{N}}$ and thus improve efficiency, we can deploy techniques for reducing the variance σ^2.

Antithetic Sampling

This technique extends readily for path dependent options. In this case for each simulated scenario $X_1, \ldots, X_M$ we also consider the antithetic scenario $-X_1, \ldots, -X_M$. A similar argument as for path independent options shows that this will result in reducing σ^2 by at least be a factor of 2 (and often more in practice) as long as the payoff is an monotone function of each X_m for $m = 1, \ldots, M$, as is the case of an up-and-out call from the example above.

Control Variates

This technique can be used when the payoff H of the option we wish to price can be approximated by the payoff H' of another simpler option which is easy to price, ideally by closed form expression. Because

$$H = H' + (H - H')$$

we could use the closed form expression to price H' and Monte Carlo simulation to price $H - H'$. This will be advantageous if the variance of $H - H'$ is smaller than that of H, as can be expected when H' is a good approximation to H. In this context H' is referred to as control variate.

For example, if H is an up-and-out call as considered above, and the barrier B is high enough, then a European call $H' = \max\{S(T) - K, 0\}$ could be used as control variate. Table 1 collects the results for an up-and-out call with $K = 100$, $B = 180$, $T = 1$ in the Black-Scholes Model with $S_0 = 100$, $\sigma = 20\%$, $r = 5\%$, discretisation of paths using $M = 100$ time steps of length $\frac{T}{M} = 0.01$ each.

	MC	MC&AS	MC&CV	MC&CV&AS
$N = 1,000$ sample paths				
μ	10.0968	10.6618	10.0081	10.2325
$S_N/\sqrt{N}$	0.4571	0.2274	0.1992	0.0982
$N = 10,000$ sample paths				
μ	10.3728	10.0925	10.1628	10.0713
$S_N/\sqrt{N}$	0.1427	0.0686	0.0548	0.0385
$N = 100,000$ sample paths				
μ	10.1551	10.1671	10.1126	10.1417
$S_N/\sqrt{N}$	0.1181	0.0582	0.0444	0.0297

Table 1: Up-and-out call price μ and standard error $S_N/\sqrt{N}$ by Monte Carlo simulation (MC) combined, respectively, with antithetic sampling (AS) and a call option as control variate (CV)