3.5 Replicating Strategy

Definition 43 A trading strategy is a pair of adapted processes $(x(t), y(t))$ representing positions in cash and stock at any time $0 \leq t \leq T$. Moreover

- The time t value of a trading strategy is defined by

$$V(t) = x(t) + S(t)y(t).$$

- The strategy is called self financing if

$$dV(t) = rx(t)dt + y(t)dS(t).$$

- A replicating strategy for a European option with payoff $h(S(T))$ and exercise time T is a self financing strategy $(x(t), y(t))$ whose value at time T matches the payoff,

$$V(T) = h(S(T)).$$

Proposition 44 *The value $D(t)$ of a European option with payoff $h(S(T))$ and exercise time T for which a replicating strategy $(x(t), y(t))$ exists must be equal to the value $V(t)$ of the strategy,*

$$D(t) = V(t)$$

for any time $t \leq T$ or else an arbitrage opportunity would exist.

Proof Exactly the same as in the discrete case. ■

For a self financing strategy $(x(t), y(t))$ we consider the discounted value

$$\tilde{V}(t) = e^{-rt}V(t).$$

By the Itô formula with $f(t,x) = e^{-rt}x$ we have

$$\begin{aligned} d\tilde{V}(t) &= -re^{-rt}V(t)dt + e^{-rt}dV(t) \\ &\qquad \text{by def. of } V(t) \text{ and self financing} \\ &= -re^{-rt}\left[x(t) + S(t)y(t)\right]dt + e^{-rt}\left[rx(t)dt + y(t)dS(t)\right] \\ &= y(t)\left[-re^{-rt}S(t)dt + e^{-rt}dS(t)\right] \\ &= y(t)d\tilde{S}(t), \end{aligned}$$

where

$$\tilde{S}(t) = e^{-rt}S(t)$$

is the discounted stock price process.

Next we compute

$$
\begin{aligned}
d\tilde{S}(t) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\
&= -re^{-rt}S(t)dt + e^{-rt}\left[\left(\mu + \frac{1}{2}\sigma^2\right)S(t)dt + \sigma S(t)dB(t)\right] \quad \text{by (3.9)} \\
&= \left(\mu + \frac{1}{2}\sigma^2 - r\right)\tilde{S}(t)dt + \sigma\tilde{S}(t)dB(t) \\
&= \sigma\tilde{S}(t)\left[\frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma}dt + dB(t)\right] \\
&= \sigma\tilde{S}(t)dB^*(t),
\end{aligned}
$$

where

$$
\begin{aligned}
B^*(t) &= \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma}t + B(t) \\
&= bt + B(t)
\end{aligned}
$$

with

$$
b = \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma}.
$$

Here $B(t)$ is Brownian motion under probability P, but of course $B^*(t)$ is not, if only because $\mathbb{E}[B^*(t)] = bt \neq 0$. However, if we could find a probability P^* such that $B^*(t)$ becomes Brownian motion under P^*, then we would be able to argue as follows: Since

$$
d\tilde{V}(t) = y(t)d\tilde{S}(t) = \sigma y(t)\tilde{S}(t)dB^*(t),
$$

which takes the form

$$
\tilde{V}(T) = \tilde{V}(0) + \int_0^T \sigma y(t)\tilde{S}(t)dB^*(t)
$$

when the integrals a duly reinstated., applying the expectation under P^* to both sides, we would then get

$$
\mathbb{E}^*\left[\tilde{V}(T)\right] = V(0)
$$

because the expectation of an Itô integral with respect to Brownian motion is zero. If $(x(t), y(t))$ is a replicating strategy, then $V(0) = D(0)$ and $V(T) = h(S(T))$, so that the last displayed equality could be written as

$$
D(0) = \mathbb{E}^*\left[e^{-rT}h(S(T))\right],
$$

providing a formula for the option price, similar to what we have seen in the discrete case.

3.6 Girsanov Theorem

Theorem 45 (Girsanov theorem simplified, constant drift only) *Let*

$$B^*(t) = bt + B(t),$$

where $B(t)$ is Brownian motion under P, and let

$$P^*(A) = \mathbb{E}\left[e^{-bB(T)-\frac{1}{2}b^2T} 1_A\right] \tag{3.10}$$

for any event $A \subset \Omega$, where the expectation on the right-hand side is taken under probability P, and where

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A, \end{cases}$$

is the indicator function of A. Then

(1) *P^* is a probability measure;*

(2) *$B^*(t)$ is Brownian motion under P^*.*

Proof (1) It follows immediately from (3.10) that P^* is a measure. (In fact (3.10) means that P^* is an absolutely continuous measure with respect to P with density $\frac{dP^*}{dP} = e^{-bB(T)-\frac{1}{2}b^2T}$.) We need to check that P^* is a probability measure, that is, $P^*(\Omega) = 1$. Indeed, by (3.10) and using the fact the $B(T) \sim N(0,T)$ under P, we obtain

$$\begin{aligned} P^*(\Omega) &= \mathbb{E}\left[e^{-bB(T)-\frac{1}{2}b^2T} 1_\Omega\right] = \mathbb{E}\left[e^{-bB(T)-\frac{1}{2}b^2T}\right] \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-bx-\frac{1}{2}b^2T} e^{-\frac{x^2}{2T}}\, dx \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{(x+bT)^2}{2T}}\, dx \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2T}}\, dy \\ &= 1. \end{aligned}$$

Here we used the density $\frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}$ of the normal distribution $N(0,T)$ and made a substitution $y = x + bT$.

(2) To show that $B^*(t)$ is a Brownian motion under P^* we need to verify that is satisfies the conditions from the definition of Brownian motion. Since $B(t)$ has stationary independent increments, continuous trajectories and $B(0) = 0$, it follows immediately that $B^*(t) = bt + B(t)$ also has stationary independent increments, continuous trajectories and $B^*(0) = 0$.

All that remains to be proved is that $B^*(t) \sim N(0,t)$ under P^* for any $0 \le t \le T$. To this end, we compute for any $a \in \mathbb{R}$

$$\begin{aligned}
P^*\{B^*(t) \le a\} &= \mathbb{E}\left[e^{-bB(T)-\frac{1}{2}b^2T}1_{\{B^*(t)\le a\}}\right] \\
&= \mathbb{E}\left[e^{-b(B(T)-B(t))-\frac{1}{2}b^2(T-t)}e^{-bB(t)-\frac{1}{2}b^2t}1_{\{bt+B(t)\le a\}}\right] \\
&= \mathbb{E}\left[e^{-b(B(T)-B(t))-\frac{1}{2}b^2(T-t)}\right]\mathbb{E}\left[e^{-bB(t)-\frac{1}{2}b^2t}1_{\{B(t)\le a-bt\}}\right].
\end{aligned}$$

Since $B(T)-b(t) \sim N(0,T-t)$, by a similar argument as in part (1) we can show that

$$\mathbb{E}\left[e^{-b(B(T)-B(t))-\frac{1}{2}b^2(T-t)}\right] = 1.$$

Thus

$$\begin{aligned}
P^*\{B^*(t) \le a\} &= \mathbb{E}\left[e^{-bB(t)-\frac{1}{2}b^2t}1_{\{B(t)\le a-bt\}}\right] \\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a-bt} e^{-bx-\frac{1}{2}b^2t}e^{-\frac{x^2}{2t}}\,dx \\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a-bt} e^{-\frac{(x+bt)^2}{2t}}\,dx \\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a} e^{-\frac{y^2}{2t}}\,dy,
\end{aligned}$$

where we have used the density of the normal distribution $N(0,t)$ and made the substitution $y = x + bt$. It follows that, indeed, $B^*(t) \sim N(0,t)$ under P^*, completing the proof. ■

The probability P^* constructed in the Girsanov theorem with

$$b = \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma}$$

will be called the risk neutral probability. This is because for the discounted stock price, which satisfies $d\tilde{S}(t) = \sigma\tilde{S}(t)dB^*(t)$, it gives

$$\tilde{S}(0) = \mathbb{E}^*\left[\tilde{S}(T)\right].$$

In a hypothetical wold in which stock price movements were governed by probability P^* (rather than the real world probability P) one wouldn't expect to gain or lose on average by investing in stock. This is usually not the case in the real world, where investors want to be compensated for taking risk by some growth in expected stock prices.

3.7 Option Price via Risk Neutral Probability

We have established that the discounted value $\tilde{V}(t) = e^{-rt}V(t)$ of a self financing strategy $(x(t), y(t))$ satisfies

$$\tilde{V}(T) = V(0) + \int_0^T y(t)\tilde{S}(t)dB^*(t).$$

We have also constructed probability P^*, called the risk neutral probability, such that $B^*(t)$ is Brownian motion under P^*. Thus, taking the expectation under P^* on both sides of the above equality, we get

$$\mathbb{E}^* \left[\tilde{V}(T)\right] = V(0)$$

because the expectation of a stochastic integral with respect to Brownian motion is zero. If the strategy replicates a European option with exercise time T and payoff of the form $h(S(T))$, then

$$V(T) = h((T)),$$

and we obtain

$$V(0) = \mathbb{E}^* \left[e^{-rT} h(S(T))\right].$$

Because the initial value $V(0)$ of a replicating strategy must math the initial value $D(0)$ of the derivative security, $V(0) = D(0)$, we have in effect proved the following very important result for option pricing in the Black-Scholes model.

Theorem 46 *If a European type derivative security with exercise time T and payoff $h(S(T))$ can be replicated by a self financing strategy, then the value of the derivative security at time* 0 *is given by*

$$D(0) = \mathbb{E}^* \left[e^{-rT} h((S(T))\right].$$

Example 47 Consider a European call with exercise time T and strike price K, so that

$$h(S(T)) = \max\left(S(T) - K, 0\right).$$

Assuming that this option can be replicated (this will be shown later), its time 0 price $C(0)$ will be

$$\begin{aligned} C(0) &= \mathbb{E}^* \left[e^{-rT} h(S(T))\right] \\ &= e^{-rT}\mathbb{E}^* \left[\max\left(S(T) - K, 0\right)\right] \\ &= e^{-rT}\mathbb{E}^* \left[\left(S(T) - K\right) 1_{S(T)>K}\right] \\ &= e^{-rT}\mathbb{E}^* \left[S(T) 1_{S(T)>K}\right] - e^{-rT} K \mathbb{E}^* \left[1_{S(T)>K}\right]. \end{aligned}$$

We can compute these two expectations using the fact that in the Black-Scholes model

$$\begin{aligned} S(T) &= S(0) e^{\mu T + \sigma B(T)} \\ &= S(0) e^{\mu T + \sigma \left(B^*(T) - \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma} T \right)} = S(0) e^{\left(r - \frac{1}{2}\sigma^2\right) T + \sigma B^*(T)} \end{aligned}$$

and $B^*(t)$ is Brownian motion under P^*. Observe that

$$S(T) > K \iff B^*(T) > -\frac{\ln \frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma}.$$

It follows that

$$
\begin{aligned}
\mathbb{E}^*\left[S(T)1_{S(T)>K}\right] &= \mathbb{E}^*\left[S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma B^*(T)}1_{B^*(T)>-\frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma}}\right] \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma}}^{\infty} S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma x}e^{-\frac{x^2}{2T}}\,dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma}}^{\infty} S(0)e^{rT}e^{-\frac{(x-\sigma)^2}{2T}}\,dx \\
&= S(0)e^{rT}\frac{1}{\sqrt{2\pi}}\int_{-\frac{\ln\frac{S(0)}{K}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{-\frac{y^2}{2}}\,dy \quad \text{substitute } y=\frac{x-\sigma}{\sqrt{T}} \\
&= S(0)e^{rT}\frac{1}{\sqrt{2\pi}}\int_{-d_1}^{\infty} e^{-\frac{y^2}{2}}\,dy \\
&= S(0)e^{rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{y^2}{2}}\,dy \\
&= S(0)e^{rT}N(d_1)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}^*\left[1_{S(T)>K}\right] &= \mathbb{E}^*\left[1_{B^*(T)>-\frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma}}\right] \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma}}^{\infty} e^{-\frac{x^2}{2T}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{-\frac{y^2}{2}}\,dy \quad \text{substitute } y=\frac{x}{\sqrt{T}} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\,dy \\
&= N(d_2)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S(0)}{K}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln\frac{S(0)}{K}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}
\end{aligned}
$$

and where

$$
N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}\,dx
$$

is the cumulative distribution function of the standard normal distribution $N(0,1)$.

As a result, we obtain the famous Black-Scholes formula for the price of a European call

$$\begin{aligned} C(0) &= e^{-rT}\mathbb{E}^*\left[S(T)1_{S(T)>K}\right] - e^{-rT}K\mathbb{E}^*\left[1_{S(T)>K}\right] \\ &= S(0)N(d_1) - e^{-rT}KN(d_2). \end{aligned}$$