Example 29 We shall compute $\int_0^T t dB(t)$. First we observe that the integrand t is Itô integrable. Then we consider the sum

$$\begin{aligned}
&\sum_{i=0}^{N-1} t_i \left(B(t_{i+1}) - B(t_i)\right) \\
&= \sum_{i=0}^{N-1} \left(t_{i+1}B(t_{i+1}) - t_i B(t_i)\right) - \sum_{i=0}^{N-1} B(t_{i+1}) \left(t_{i+1} - t_i\right) \\
&= TB(T) - \sum_{i=0}^{N-1} B(t_{i+1}) \left(t_{i+1} - t_i\right).
\end{aligned}$$

In the limit as $N \to \infty$ we obtain

$$\int_0^T t dB(t) = TB(T) - \int_0^T B(t) dt.$$

Example 30 Next we shall compute $\int_0^T B(t)^2 dB(t)$. The integrand $B(t)^2$ is adapted and

$$\int_0^T \mathbb{E}\left[\left(B(t)^2\right)^2\right] dt = \int_0^T 3t^2 dt = T^3 < \infty,$$

so $B(t)^2$ is Itô integrable. The sum approximating the stochastic integral $\int_0^T B(t)^2 dB(t)$ can be written as follows:

$$\begin{aligned}
&\sum_{i=0}^{N-1} B(t_i)^2 \left(B(t_{i+1}) - B(t_i)\right) \\
&= \frac{1}{3} \sum_{i=0}^{N-1} \left(B(t_{i+1})^3 - B(t_i)^3\right) - \sum_{i=0}^{N-1} B(t_i) \left(B(t_{i+1}) - B(t_i)\right)^2 \\
&\quad - \frac{1}{3} \sum_{i=0}^{N-1} \left(B(t_{i+1}) - B(t_i)\right)^3 \\
&= \frac{1}{3} B(T)^3 - \sum_{i=0}^{N-1} B(t_i) \left(t_{i+1} - t_i\right) - \sum_{i=0}^{N-1} B(t_i) \left[\left(B(t_{i+1}) - B(t_i)\right)^2 - \left(t_{i+1} - t_i\right)\right] \\
&\quad - \frac{1}{3} \sum_{i=0}^{N-1} \left(B(t_{i+1}) - B(t_i)\right)^3.
\end{aligned}$$

Clearly,

$$\sum_{i=0}^{N-1} B(t_i) \left(t_{i+1} - t_i\right) \to \int_0^T B(t) dt \quad \text{as } N \to \infty.$$

We shall show that

$$\sum_{i=0}^{N-1} B(t_i)\left[(B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right] \to 0 \quad \text{in } L^2 \text{ as } N\to\infty,$$

$$\sum_{i=0}^{N-1} (B(t_{i+1})-B(t_i))^3 \to 0 \quad \text{in } L^2 \text{ as } N\to\infty.$$

To this end we consider the second moments

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=0}^{N-1} B(t_i)\left[(B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right]\right)^2\right] \\
&=\sum_{i=0}^{N-1}\mathbb{E}\left[\left(B(t_i)\left[(B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right]\right)^2\right] \\
&=\sum_{i=0}^{N-1}\mathbb{E}\left[B(t_i)^2\right]\mathbb{E}\left[\left((B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right)^2\right] \\
&=\sum_{i=0}^{N-1} t_i\mathbb{E}\left[(B(t_{i+1})-B(t_i))^4-2\,(B(t_{i+1})-B(t_i))^2\,(t_{i+1}-t_i)+(t_{i+1}-t_i)^2\right] \\
&=\sum_{i=0}^{N-1} t_i\left[3\,(t_{i+1}-t_i)^2-2\,(t_{i+1}-t_i)^2+(t_{i+1}-t_i)^2\right] \\
&=2\sum_{i=0}^{N-1} t_i\left[(t_{i+1}-t_i)^2\right]=2\sum_{i=0}^{N-1}\frac{T}{N}i\left(\frac{T}{N}\right)^2 \\
&=\frac{2T^3}{N^3}\frac{N(N-1)}{2}=\frac{T^3(N-1)}{N^2}\to 0 \quad \text{as } N\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=0}^{N-1}(B(t_{i+1})-B(t_i))^3\right)^2\right] \\
&=\sum_{i=0}^{N-1}\mathbb{E}\left[(B(t_{i+1})-B(t_i))^6\right] \\
&=6\sum_{i=0}^{N-1}(t_{i+1}-t_i)^3=6\sum_{i=0}^{N-1}\left(\frac{T}{N}\right)^3=\frac{6T^3}{N^2}\to 0 \quad \text{as } N\to\infty.
\end{aligned}$$

It follows that

$$\int_0^T B(t)^2 dB(t)=\frac{1}{3}B(T)^3-\int_0^T B(t)dt.$$

These examples are instances of a general pattern, captured in the following very important result.

Theorem 31 (Itô formula, simplified) *Let $f(t,x)$ be a function with continuous partial derivatives*

$$f'_t(t,x), f'_x(t,x), f''_{xx}(t,x)$$

and such that $f'_x(t,B(t))$ is Itô integrable. Then

$$\begin{aligned} & f(T,B(T)) \\ & = f(0,B(0)) + \int_0^T \left(f'_t(t,B(t)) + \frac{1}{2} f''_{xx}(t,B(t)) \right) dt + \int_0^T f'_x(t,B(t)) dB(t). \end{aligned} \tag{3.5}$$

Remark 32 It is customary to write expressions line the Itô formula (3.5) in shorthand notation as follows:

$$df(t,B(t)) = \left(f'_t(t,B(t)) + \frac{1}{2} f''_{xx}(t,B(t)) \right) dt + f'_x(t,B(t)) dB(t).$$

This is called Itô differential notation, and it is purely formal but very convenient. Whenever we write down an expression like that, the reader is expected to imagine the integrals $\int_0^T$ in the right places so that formula (3.5) is recovered.

Before proving this theorem, let us see how it can be used to evaluate the same stochastic integrals that were computed in Examples 28, 29 and 30 directly from the definition.

Example 33 We want to compute $\int_0^T B(t)dB(t)$. To this end we take $f(t,x) = x^2$, so that $f'_t(t,x) = 0$, $f'_x(t,x) = 2x$, $f''_{xx}(t,x) = 2$. From the Itô formula (3.5) we get

$$B(T)^2 = B(0)^2 + \int_0^T 2dt + \int_0^T 2B(t)dB(t),$$

so that

$$\int_0^T B(t)dB(t) = \frac{1}{2} B(T)^2 - \frac{1}{2} T.$$

In the shorthand differential notation this can be written as

$$B(t)dB(t) = \frac{1}{2} d\left[B(t)^2 \right] - \frac{1}{2} dt.$$

Example 34 Next we shall compute $\int_0^T tdB(t)$. Take $f(t,x) = tx$, so $f'_t(t,x) = x$, $f'_x(t,x) = t$, $f''_{xx}(t,x) = 0$. The Itô formula then gives

$$TB(T) = 0B(0) + \int_0^T B(t)dt + \int_0^T tdB(t).$$

It follows that

$$\int_0^T tdB(t) = TB(T) - \int_0^T B(t)dt.$$

In differential notation this becomes

$$tdB(t) = d\left[TB(T) \right] - B(t)dt.$$

Example 35 Finally, we compute $\int_0^T B(t)^2 dB(t)$. Taking $f(t,x) = x^3$, we have $f_t'(t,x) = 0$, $f_x'(t,x) = 3x^2$, $f_{xx}''(t,x) = 6x$ and

$$B(T)^3 = B(0)^3 + \int_0^T 3B(T)dt + \int_0^T 3B(t)^2 dB(t).$$

As a result,

$$\int_0^T B(t)^2 dB(t) = \frac{1}{3}B(T)^3 - \int_0^T B(T)dt,$$

or in differential notation

$$B(t)^2 dB(t) = \frac{1}{3} d\left[B(T)^3\right] - B(T)dt.$$

Proof outline of Theorem 31 We shall prove the Itô formula (3.5) in the case when $f(t,x)$ has bounded derivatives of all orders and can be expanded into a Taylor series about any point (t_0, x_0):

$$\begin{aligned} f(t,x) &= f(t_0,x_0) + f_t'(t_0,x_0)(t-t_0) + f_x'(t_0,x_0)(x-x_0) \\ &\qquad + \frac{1}{2} f_{xx}''(t_0,x_0)(x-x_0)^2 + \cdots \\ &= \sum_{m,n\geq 0} \frac{1}{m!n!} f_{t\,x}^{(m)(n)}(t_0,x_0)(t-t_0)^m (x-x_0)^n, \end{aligned}$$

where $f_{t\,x}^{(m)(n)}(t_0,x_0)$ denotes the partial derivative of order m with respect to t and order n with respect to x. The Itô formula for a more general function $f(t,x)$ can then be obtained by approximation by functions form this class.

From the Taylor expansion we obtain

$$\begin{aligned} & f(T,B(T)) - f(0,B(0)) \\ &= \sum_{i=0}^{N-1} \left(f(t_{i+1},B(t_{i+1})) - f(t_i,B(t_i))\right) \\ &= \sum_{i=0}^{N-1} f_t'(t_i,B(t_i))(t_{i+1}-t_i) + \sum_{i=0}^{N-1} f_x'(t_i,B(t_i))(B(t_{i+1})-B(t_i)) \\ &\quad + \frac{1}{2}\sum_{i=0}^{N-1} f_{xx}''(t_i,B(t_i))(B(t_{i+1})-B(t_i))^2 \\ &\quad + \cdots \end{aligned}$$

which can be written as

$$
\begin{aligned}
&f(T,B(T)) - f(0,B(0)) \\
&= \sum_{i=0}^{N-1} f'_t(t_i,B(t_i))(t_{i+1}-t_i) + \sum_{i=0}^{N-1} f'_x(t_i,B(t_i))(B(t_{i+1})-B(t_i)) \\
&\quad + \frac{1}{2}\sum_{i=0}^{N-1} f''_{xx}(t_i,B(t_i))(t_{i+1}-t_i) \\
&\quad + \frac{1}{2}\sum_{i=0}^{N-1} f''_{xx}(t_i,B(t_i))\left[(B(t_{i+1})-B(t_i))^2 - (t_{i+1}-t_i)\right] \\
&\quad + \dots
\end{aligned}
$$

Clearly,

$$\sum_{i=0}^{N-1} f'_t(t_i,B(t_i))(t_{i+1}-t_i) \to \int_0^T f'_t(t,B(t))dt, \tag{3.6}$$

$$\sum_{i=0}^{N-1} f'_x(t_i,B(t_i))(B(t_{i+1})-B(t_i)) \to \int_0^T f'_x(t,B(t))dB(t), \tag{3.7}$$

$$\sum_{i=0}^{N-1} f''_{xx}(t_i,B(t_i))(t_{i+1}-t_i) \to \int_0^T f''_{xx}(t,B(t))dt \tag{3.8}$$

as $N \to \infty$. If we can show that the remaining terms converge to 0 in L^2, we will have proved the Itô formula (3.5). We start with

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=0}^{N-1} f''_{xx}(t_i,B(t_i))\left[(B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right]\right)^2\right] \\
&= \sum_{i=0}^{N-1}\mathbb{E}\left[f''_{xx}(t_i,B(t_i))\left[(B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right]^2\right] \\
&= \sum_{i=0}^{N-1}\mathbb{E}\left[f''_{xx}(t_i,B(t_i))\right]\mathbb{E}\left[\left[(B(t_{i+1})-B(t_i))^2-(t_{i+1}-t_i)\right]^2\right] \\
&\leq \cdots
\end{aligned}
$$

Since f''_{xx} is bounded, say by C, this can be estimates as

$$\cdots \leq C \sum_{i=0}^{N-1} \mathbb{E}\left[\left[(B(t_{i+1}) - B(t_i))^2 - (t_{i+1} - t_i)\right]^2\right]$$

$$= C \sum_{i=0}^{N-1} \mathbb{E}\left[(B(t_{i+1}) - B(t_i))^4 - 2(B(t_{i+1}) - B(t_i))^2(t_{i+1} - t_i) + (t_{i+1} - t_i)^2\right]$$

$$= C \sum_{i=0}^{N-1} \left[3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2\right]$$

$$= 2C \sum_{i=0}^{N-1} (t_{i+1} - t_i)^2 = 2C \sum_{i=0}^{N-1} \left(\frac{T}{N}\right)^2 = \frac{2CT^2}{N} \to 0 \quad \text{as } N \to \infty.$$

Next, we take a general term of the form

$$\sum_{i=0}^{N-1} \frac{1}{m!n!} f^{(m)(n)}_{t\,x}(t_i, B(t_i))(t_{i+1} - t_i)^m (B(t_{i+1}) - B(t_i))^n$$

and investigate when this will converge to 0 in L^2. We have

$$\mathbb{E}\left[\sum_{i=0}^{N-1} \left(f^{(m)(n)}_{t\,x}(t_i, B(t_i))(t_{i+1} - t_i)^m (B(t_{i+1}) - B(t_i))^n\right)^2\right]$$

$$\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mathbb{E}\left[(t_{i+1} - t_i)^m (B(t_{i+1}) - B(t_i))^n (t_{j+1} - t_j)^m (B(t_{j+1}) - B(t_j))^n\right]$$

$$= C(t_{i+1} - t_i)^m (t_{j+1} - t_j)^m \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mathbb{E}\left[(B(t_{i+1}) - B(t_i))^n (B(t_{j+1}) - B(t_j))^n\right]$$

$$= \ldots$$

If n is even, this gives

$$\ldots = C(t_{i+1} - t_i)^m (t_{j+1} - t_j)^m$$
$$\times \left(\sum_{i=0}^{N-1} \mathbb{E}\left[(B(t_{i+1}) - B(t_i))^{2n}\right] + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} \mathbb{E}\left[(B(t_{i+1}) - B(t_i))^n\right] \mathbb{E}\left[(B(t_{j+1}) - B(t_j))^n\right]\right)$$

$$\leq C'(t_{i+1} - t_i)^m (t_{j+1} - t_j)^m \left(\sum_{i=0}^{N-1} (t_{i+1} - t_i)^n + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} (t_{i+1} - t_i)^{n/2} (t_{j+1} - t_j)^{n/2}\right)$$

$$= C'(t_{i+1} - t_i)^m (t_{j+1} - t_j)^m (t_{i+1} - t_i)^n \left(N + N(N-1)\right)$$

$$= C' \left(\frac{T}{N}\right)^m \left(\frac{T}{N}\right)^m \left(N \left(\frac{T}{N}\right)^n + N(N-1) \left(\frac{T}{N}\right)^{n/2} \left(\frac{T}{N}\right)^{n/2}\right)$$

$$= \frac{C'T^{2m+n}}{N^{2m+n-2}} \to 0 \quad \text{as } N \to \infty$$

if $2m+n-2>0$, which is satisfied for all $m \geq 0$ and all even $n \geq 0$ except for $m=0, n=2$ and $m=1, n=0$. On the other hand, if n is odd, then the above expression can be written as

$$\begin{aligned}
... &= C \sum_{i=0}^{N-1} \mathbb{E}\left[(t_{i+1}-t_i)^{2m}(B(t_{i+1})-B(t_i))^{2n}\right] \\
&= C(t_{i+1}-t_i)^{2m} \sum_{i=0}^{N-1} \mathbb{E}\left[(B(t_{i+1})-B(t_i))^{2n}\right] \\
&= C'(t_{i+1}-t_i)^{2m} \sum_{i=0}^{N-1} (t_{i+1}-t_i)^{n} \\
&= C' \left(\frac{T}{N}\right)^{2m} \sum_{i=0}^{N-1} \left(\frac{T}{N}\right)^{n} \\
&= \frac{C'T^{2m+n}}{N^{2m+n-1}} \to 0 \quad \text{as } N \to \infty
\end{aligned}$$

if $2m+n-1>0$, which is satisfied for all $m \geq 0$ and all odd $n \geq 0$ except for $m=0, n=1$. This shows that only the terms (3.6), (3.7), (3.8) give non-zero contribution in the limit. ■

3.4 Itô Processes

We can see that the Itô formula (3.5) produces random processes which are of the form:

constant + integral with respect to dt + integral with respect to $dB(t)$

This motivates the following definition.

Definition 36 We call a random process $X(t)$ and Itô process if it can be expressed in the form

$$X(T) = X(0) + \int_0^T a(t)dt + \int_0^T b(t)dB(T),$$

where $a(t), b(t)$ are adapted processes such that

$$\int_0^T \mathbb{E}\left[|a(t)|\right] dt < \infty, \quad \int_0^T \mathbb{E}\left[b(t)^2\right] dt < \infty.$$

Example 37 $B(t)$ and t are obviously Itô processes. We know that

$$\begin{aligned}
d\left[B(t)^2\right] &= 2B(t)dB(t) + dt, \\
d\left[tB(t)\right] &= tdB(t) + B(t)dt, \\
d\left[B(t)^3\right] &= 3B(t)^2dB(t) + 3B(t)dt,
\end{aligned}$$

do that $B(t)^2, tB(t), B(t)^3$ are also Itô processes.

Example 38 We shall show that the stock price

$$S(t) = S(0)e^{\mu t+\sigma B(t)}$$

in the Black-Scholes model is an Itô process. Taking $f(t,x) = S(0)e^{\mu t+\sigma x}$, we have $S(t) = f(t, B(t))$. Since

$$f_t'(t,x) = \mu f(t,x), \quad f_x'(t,x) = \sigma f(t,x), \quad f_{xx}''(t,x) = \sigma^2 f(t,x),$$

by the Itô formula we have

$$dS(t) = \left(\mu(t) + \frac{1}{2}\sigma^2\right) S(t)dt + \sigma S(t)dB(t).$$

This shows that $S(t)$ is an Itô process.

The last equation is an example of a stochastic differential equation (SDE). We can say that stock price $S(t)$ in the Black-Scholes model satisfies this SDE.

Definition 39 We say that a random process $Y(t)$ is Itô integrable with respect to an Itô process

$$X(t) = X(0) + \int_0^T a(t)dt + \int_0^T b(t)dB(t)$$

if

- $Y(t)$ is adapted,
- $\int_0^T \mathbb{E}\left[|Y(t)a(t)|\right] dt < \infty$,
- $\int_0^T \mathbb{E}\left[|Y(t)b(t)|^2\right] dt < \infty$.

Then the Itô integral of $Y(t)$ with respect to $X(t)$ is defined by

$$\int_0^T Y(t)dX(t) = \int_0^T Y(t)a(t)dt + \int Y(t)b(t)dB(t).$$

Remark 40 Because $S(t)$ is an Itô process, see Example 38, we can now go back to the problem of defining a self financing strategy $(x(t), y(t))$ in the Black-Scholes model by means of condition (3.2), that is,

$$V(T) = V(0) + r\int_0^T x(t)dt + \int_0^T y(t)dS(t).$$

The integral $\int_0^T y(t)dS(t)$ now makes sense in view of Definition 39.

Finally, we state the Itô formula in its general form, with Brownian motion $B(t)$ in $f(t, B(t))$ replaced by a general Itô process $X(t)$.

Theorem 41 (Itô formula) *Let*

$$X(t) = X(0) + \int_0^T a(t)dt + \int_0^T b(t)dB(t)$$

be an Itô process and let $f(t,x)$ be a function with continuous partial derivatives

$$f'_t(t,x), f'_x(t,x), f''_{xx}(t,x)$$

and such that $b(t)f'_x(t,X(t))$ is Itô integrable. Then

$$\begin{aligned}
&f(T,X(T)) \\
&= f(0,X(0)) + \int_0^T \left(f'_t(t,X(t)) + \frac{1}{2}b(t)^2 f''_{xx}(t,X(t)) \right) dt + \int_0^T f'_x(t,X(t))dX(t) \\
&= f(0,X(0)) + \int_0^T \left(f'_t(t,X(t)) + a(t)f'_x(t,X(t)) + \frac{1}{2}b(t)^2 f''_{xx}(t,X(t)) \right) dt \\
&\qquad + \int_0^T b(t)f'_x(t,X(t))dB(t).
\end{aligned}$$

Remark 42 In the shorthand differential notation this formula looks like this:

$$\begin{aligned}
df(t,X(t)) &= \left(f'_t(t,X(t)) + \frac{1}{2}b(t)^2 f''_{xx}(t,X(t)) \right) dt + f'_x(t,X(t))dX(t) \\
&= \left(f'_t(t,X(t)) + a(t)f'_x(t,X(t)) + \frac{1}{2}b(t)^2 f''_{xx}(t,X(t)) \right) dt \\
&\qquad + b(t)f'_x(t,X(t))dB(t).
\end{aligned}$$

The proof of this theorem will be omitted. It is similar to that of Theorem 31, though somewhat more complicated.