3 Black-Scholes Model

In this section we are going to construct a continuous time model of stock prices $S(t)$ for $0 \leq t$.

In discrete time we used the simple return

$$\frac{S(t+\delta) - S(t)}{S(t)}$$

to describe the dynamics of stock prices over a time period from t to $t+\delta$. In the binary model this return was a random variable taking two possible values u, d.

In continuous time it will be much more convenient to use the logarithmic return (briefly, log return)

$$\ln \frac{S(t)}{S(s)}$$

for any $0 \leq s \leq t$. This is because log returns are additive: for any $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$

$$\ln \frac{S(t_n)}{S(t_0)} = \ln \frac{S(t_1)}{S(t_0)} + \ln \frac{S(t_2)}{S(t_1)} + \cdots + \ln \frac{S(t_n)}{S(t_{n-1})}. \tag{3.1}$$

The continuous time model of stock prices can be constructed by adopting a number of natural assumptions:

(A) Log returns on stock prices over disjoint time intervals should be independent. That is, for any $0 \leq v \leq u \leq s \leq t$

$$\ln \frac{S(t)}{S(s)} \quad \text{and} \quad \ln \frac{S(u)}{S(v)}$$

should be independent random variables.

In a model satisfying this assumption future changes in stock prices have no memory of past changes.

(B) For any $0 \leq t, s$ the probability distribution of the log return $\ln \frac{S(t+s)}{S(t)}$ should depend only on s but not on t.

According to this assumption, market conditions affecting the growth of stock prices do not change with time: $\ln \frac{S(s)}{S(0)}$ will have the same probability distribution as $\ln \frac{S(t+s)}{S(t)}$ for any $0 \leq t$.

(C) The function $t \mapsto S(t)$ is continuous with probability 1.

This condition means that the stock price does not have jumps.

A consequence of condition (B) is that for each $0 \leq s, t$ the expectation of the log return can be written as

$$\mathbb{E} \ln \frac{S(t+s)}{S(t)} = f(s)$$

for some function f. Combined with additivity (3.1), this gives

$$\begin{aligned} f(s+u) &= \mathbb{E}\ln\frac{S(t+s+u)}{S(t)} \\ &= \mathbb{E}\ln\frac{S(t+s)}{S(t)} + \mathbb{E}\ln\frac{S(t+s+u)}{S(t+s)} = f(s)+f(u) \end{aligned}$$

for all $0 \leq s,t,u$. From (C) it can be deduced that f is continuous. We can then solve the functional equation $f(s+u) = f(s)+f(u)$ to find that f is linear, $f(s) = \mu s$ for some $\mu \in \mathbb{R}$. It follows that for all $0 \leq s,t$

$$\mathbb{E}\ln\frac{S(t+s)}{S(t)} = \mu s.$$

In a similar way, from (B) we know that for each $0 \leq s,t$ the variance of the log return can be written as

$$\text{Var}\ \ln\frac{S(t+s)}{S(t)} = g(s)$$

for some function g. Using the independence assumed in (A) and additivity (3.1), we have

$$\begin{aligned} g(s+u) &= \text{Var}\ \ln\frac{S(t+s+u)}{S(t)} \\ &= \text{Var}\ \ln\frac{S(t+s)}{S(t)} + \text{Var}\ \ln\frac{S(t+s+u)}{S(t+s)} = g(s)+g(u) \end{aligned}$$

for all $0 \leq s,t,u$. Once again, we can get from (C) that g is a continuous function, and then it must be linear, $g(s) = \sigma^2 s$ for some $0 \leq \sigma \in \mathbb{R}$. The coefficient σ^2 can be chosen in this form because $g(s) \geq 0$ for all $s \geq 0$. For a risky asset the variance should be non-zero, so in fact $0 < \sigma$.

(D) Log returns have normal probability distribution,

$$\ln\frac{S(t+s)}{S(t)} \sim N(\mu s, \sigma^2 s)$$

for any $0 \leq s,t$.

Note that the expectation μs and variance $\sigma^2 s$ are already determined by (A), (B), (C). The motivation for assumption (D) comes to from market data: Historical log returns fit the normal distribution reasonably well (though not perfectly). Another argument in favour of assumption (D) is based on the Central Limit Theorem: If the behaviour of stocks is influenced by a great number of independent factors, this should result in normal distribution.

Assumptions (A), (B), (C), (D) characterise the famous Black-Scholes model. They are the subject of much discussion, and alternatives are considered, but the Black-Scholes model remains extremely important both historically and as a benchmark for other models.

Remark 21 A random variable with normal distribution $N(m, s^2)$ with mean m and variance s^2 is characterised by the probability density

$$\frac{1}{\sqrt{2\pi s^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

In addition to a stock, the Black-Scholes model involves a risk free asset, say cash. In continuous time it proves convenient to describe risk free growth in terms of continuous compounding. Thus we take $r \geq 0$ to be the continuously compounded rate of interest. An initial cash amount $B(0)$ invested at that rate will grow to

$$B(t) = e^{rt} B(0).$$

In other words,

$$rt = \ln \frac{B(t)}{B(0)}$$

is the log return on the cash investment.

3.1 Brownian Motion

The log return between times 0 and $t \geq 0$ can be written as

$$\ln \frac{S(t)}{S(0)} = \mu t + \sigma B(t),$$

which defines a collection of random variables $(B(t))_{t \geq 0}$, in other words a stochastic process, with the following properties:

1. $B(0) = 0$.
2. The paths $t \to B(t)$ are continuous with probability 1.
3. Independent increments:

   $$B(u) - B(v), B(t) - B(s)$$

   are independent for any $0 \leq v \leq u \leq s \leq t$.
4. Normally distributed stationary increments:

   $$B(t) - B(s) \sim N(0, t - s)$$

   for any $0 \leq s \leq t$.

Definition 22 A random process $(B(t))_{t\geq 0}$ satisfying conditions 1, 2, 3, 4 is called Brownian motion (or Wiener process).

In terms of Brownian motion, the stock price can be written as

$$S(t) = S(0)e^{\mu t+\sigma B(t)}.$$

This expression is called geometric Brownian motion, and it is the basic formula describing stock price dynamics in the Black-Scholes model.

Since the logarithm of $S(t)$ is normally distributed, stock price in the Black-Scholes model is said to have the log normal distribution.

3.2 Itô Integral

A motivation for this new type of integral that we are going to introduce comes, in particular, form the need to characterise self financing investment strategies in continuous time.

When working in discrete time, we characterised self financing strategies consisting of cash and bonds by condition (2.4):

$$(1+r)x_n + S(n)y_n = x_{n+1} + S(n)y_{n+1}.$$

However, to extend to continuous time, it is better to write this in a different way, in terms of increments in the value of a self financing strategy:

$$\begin{aligned}
\Delta V(n) &= V(n) - V(n-1)\\
&= ((1+r)x_n + S(n)y_n) - ((1+r)x_{n-1} + S(n-1)y_{n-1})\\
&= ((1+r)x_n + S(n)y_n) - (x_n + S(n-1)y_n) \qquad \text{by (2.4)}\\
&= rx_n + (S(n) - S(n-1))y_n\\
&= rx_n\Delta n + y_n\Delta S(n),
\end{aligned}$$

where $\Delta n = n - (n-1) = 1$ can be viewed as an increment in time, and $\Delta S(n) = S(n) - S(n-1)$ as an increment in stock price. It follows that for each N

$$\begin{aligned}
V(N) &= V(0) + \sum_{n=1}^{N} \Delta V(n)\\
&= V(0) + r\sum_{n=1}^{N} x_n\Delta n + \sum_{n=1}^{N} y_n\Delta S_n.
\end{aligned}$$

This equality is in fact equivalent to the self financing condition (2.4) in the discrete case.

A natural way to extend this self financing condition to continuous time would be to replace the sums by integrals. A self financing strategy $(x(t), y(t))$ consisting of cash and stock should satisfy.

$$V(T) = V(0) + r\int_0^T x(t)dt + \int_0^T y(t)dS(t),$$

where

$$V(t) = x(t) + S(t)y(t)$$

is the value of the strategy.

However, the integral $\int_0^T y(t)dS(t)$ causes some serious difficulties and needs to be defined and handled very carefully. This will lead us to the notion of Itô integrals. We shall begin with simpler Itô integrals of the form $\int_0^T X(t)dB(t)$, and then work our way to understand more complicated ones like $\int_0^T y(t)dS(t)$.

On the other hand, the integral $\int_0^T x(t)dt$ is the familiar integral with respect to dt and causes no difficulties whatsoever. If $y(t) \equiv 0$, i.e. the portfolio consists of cash only, then clearly $V(T) = x(t)$ and we get

$$V(T) = r\int_0^T V(t)dt$$

or, on differentiating with respect to T,

$$\frac{dV(t)}{dt} = rV(t)$$

so that

$$V(t) = V(0)e^{rt},$$

which is consistent with our adopted convention that a cash investment grows according to the continuously compounded rate of interest r.

3.2.1 Definition of Itô Integral

We want to define the integral $\int_0^T X(t)dB(t)$. The idea here is to look for a class of random processes $X(t)$ for which $\int_0^T X(t)dB(t)$ can be obtained as a limit of approximating sums of the form

$$\sum_{i=0}^{N-1} X(t_i)(B(t_{i+1}) - B(t_i)),$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a division of the interval $[0, T]$ into N equal subintervals. It is important here that the time argument in $X(t_i)$ is taken to be the left end t_i of the interval $[t_i, t_{i+1}]$.

Consider the first and second moments of this approximating sum. If $X(t_i)$

is independent of $(B(t_{i+1}) - B(t_i))$, we find that

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{N-1} X(t_i)(B(t_{i+1}) - B(t_i))\right] &= \sum_{i=0}^{N-1} \mathbb{E}\left[X(t_i)\right]\mathbb{E}\left[B(t_{i+1}) - B(t_i)\right] \\ &= 0, \\ \mathbb{E}\left[\left(\sum_{i=0}^{N-1} X(t_i)(B(t_{i+1}) - B(t_i))\right)^2\right] &= \sum_{i=0}^{N-1} \mathbb{E}\left[X(t_i)^2\right]\mathbb{E}\left[(B(t_{i+1}) - B(t_i))^2\right] \\ &= \sum_{i=0}^{N-1} \mathbb{E}\left[X(t_i)^2\right](t_{i+1} - t_i).\end{aligned}$$

The sum on the right-hand side of the expression for the second moment can be viewed as an approximating sum for the integral $\int_0^T \mathbb{E}\left[X(t)^2\right] dt$. Thus, if the integral exists (in the sense that it has a finite value), then

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{i=0}^{N-1} X(t_i)(B(t_{i+1}) - B(t_i))\right)^2\right] = \int_0^T \mathbb{E}\left[X(t)^2\right] dt. \tag{3.2}$$

Example 23 *1.* Take $X(t) = B(t)$. Then

$$\int_0^T \mathbb{E}\left[X(t)^2\right] dt = \int_0^T \mathbb{E}\left[B(t)^2\right] dt = \int_0^T t dt = \frac{T^2}{2}.$$

Example 24 Take $X(t) = \frac{B(t)}{t}$. Then

$$\int_0^T \mathbb{E}\left[X(t)^2\right] dt = \int_0^T \mathbb{E}\left[\frac{B(t)^2}{t^2}\right] dt = \int_0^T \frac{1}{t} dt = +\infty.$$

Going back to (3.2), we see that the $X(t_i)$'s have to be independent of the increments $B(t_{i+1}) - B(t_i)$ for all choices of t_i, where $i = 0, \ldots, N-1$. This will be so if $X(t)$ is a so-called adapted process, defined as follows.

Definition 25 A random process $(X(t))_{t\geq 0}$ is called adapted if for each $t \geq 0$ we can express $X(t)$ as a function of $B(s_1), \ldots, B(s_n)$ for some $s_1, \ldots, s_n \in [0, t]$, or as a limit of such functions.

We are now ready to state the definition of Itô integrable processes and the Itô integral.

Definition 26 A random process $(X(t))_{t\geq 0}$ is called Itô integrable on the interval $[0, T]$ if

1. $(X(t))_{t\geq 0}$ is adapted,
2. $\int_0^T \mathbb{E}\left[X(t)^2\right] dt < \infty$.

The Itô integral is then defined as the random variable

$$\int_0^T X(t)dB(t) = \lim_{n\to\infty} \sum_{i=1}^{N-1} X(t_i)\left(B(t_{i+1}) - B(t_i)\right),$$

where the limit is taken in L^2 norm.

Remark 27 We say that Y is the limit in L^2 norm of a sequence of random variables Y_n whenever

$$\lim_{n\to\infty} \mathbb{E}\left[(Y_n - Y)^2\right] = 0.$$

3.3 Itô Formula

Suppose that $f(t), g(t)$ are differentiable functions. Then

$$\begin{aligned} f(g(T)) &= f(g(0)) + \int_0^T f'(g(t))g'(t)dt \\ &= f(g(0)) + \int_0^T f'(g(t))dg(t), \end{aligned}$$

which is the integrated form of the chain rule

$$\frac{df(g(t))}{dt} = f'(g(t))g'(t).$$

However, if we try to substitute Brownian motion for $g(t)$, then it turns out that, in general

$$f(B(t)) \neq f(B(0)) + \int_0^T f'(B(t))dB(t). \tag{3.3}$$

Example 28 Take $f(x) = x^2$. Then

$$\begin{aligned} f(B(t_{i+1})) - f(B(t_i)) &= B(t_{i+1})^2 - B(t_i)^2 \\ &= 2B(t_i)\left(B(t_{t+1}) - B(t_i)\right) + \left(B(t_{i+1}) - B(t_i)\right)^2, \end{aligned}$$

so that

$$B(T)^2 - B(0)^2 = 2\sum_{i=1}^{N-1} B(t_i)\left(B(t_{t+1}) - B(t_i)\right) + \sum_{i=1}^{N-1}\left(B(t_{i+1}) - B(t_i)\right)^2$$

The L^2 limit as $N \to \infty$ of the first sum is the Itô integral $\int_0^T B(t)dB(t)$. What is the limit in L^2 norm of the second sum? We have

$$\mathbb{E}\left[\sum_{i=1}^{N-1}\left(B(t_{i+1}) - B(t_i)\right)^2\right] = \sum_{i=1}^{N-1}\mathbb{E}\left[\left(B(t_{i+1}) - B(t_i)\right)^2\right] = \sum_{i=1}^{N-1}(t_{i+1} - t_i) = T,$$

which suggests that T should is the limit also in L^2 norm. To verify this, we need to check that

$$\mathbb{E}\left[\left(\sum_{i=1}^{N-1}(B(t_{i+1})-B(t_i))^2-T\right)^2\right]\to 0,$$

which will be done as an exercise. We then get

$$B(T)^2=2\int_0^T B(t)dB(t)+T.$$

This shows that in general we cannot have an equality in (3.3).

As a byproduct we have computed our first Itô integral:

$$\int_0^T B(t)dB(t)=\frac{1}{2}B(T)^2-\frac{1}{2}T.$$