2 Multi Period Binary Model

In Section 1 we introduced the single period binary model with trading dates $t = 0, T$. This model involved cash as a risk free asset with rate of return r and a stock as a risky asset. The return on the stock is a random variable with two values $u > d$, where

$$K = \frac{S(T) - S(0)}{S(0)} = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1 - p, \end{cases},$$

so that $S^u(1) = S(0)(1 + u)$ and $S^d(1) = S(0)(1 + d)$. The necessary condition (1.4) for the lack of arbitrage established in the proof of Proposition 10 can then be written as

$$d < r < u$$

and formula (1.3) for the risk neutral probability becomes

$$p^* = \frac{r - d}{u - d}.$$

We shall now consider a model with N periods of equal length $\delta = \frac{T}{N}$ between times 0 and $T > 0$, corresponding to $N+1$ trading dates $0, \delta, 2\delta, \ldots, (N-1)\delta, T$. The rate of return for a risk free cash investment over each of the N time periods of length $\delta = \frac{T}{N}$ will be r, so that a unit of cash invested at this rate at time 0 will be worth $(1+r)^n$ at time $n\delta$. To model the stock prices, we take a sequence of independent identically distributed (iid) random variables $K_1, \ldots, K_N$, such that for each $n = 1, \ldots, N$

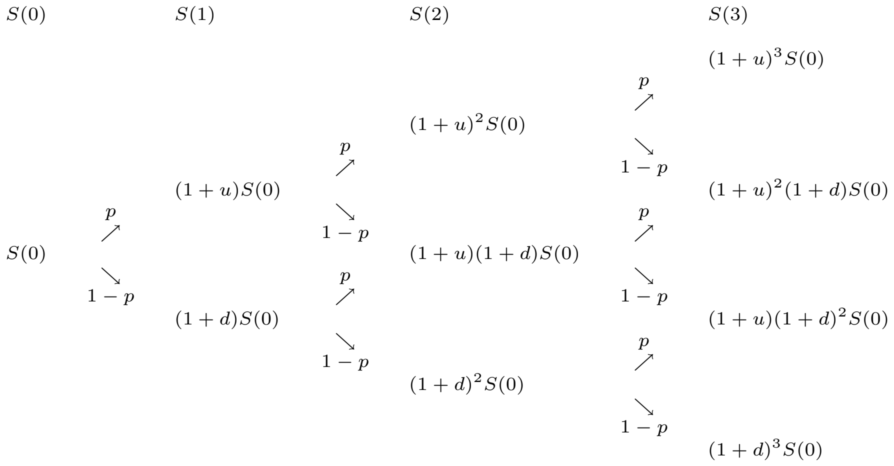
$$K_n = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1 - p. \end{cases}$$

and construct the stock prices by

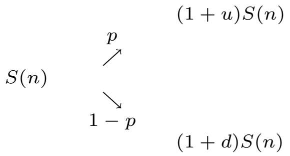
$$\begin{aligned} S(\delta) &= (1 + K_1)S(0), \\ S(2\delta) &= (1 + K_2)S(\delta), \\ &\vdots \\ S(T) &= (1 + K_N)S((N - 1)\delta). \end{aligned}$$

To simplify the notation, we shall often write $S(n)$ instead of $S(n\delta)$ for $n = 0, 1, \ldots, N$.

The stock prices can be represented by a tree. For example, for $N = 3$



Every subtree of the form



has the same structure as the single period tree in Section 1.

The probability space behind this model can be identified with

$$\Omega = \{\mathrm{u}, \mathrm{d}\}^N,$$

which consists of N-element sequences $\omega = \omega_1\omega_2 \cdots \omega_N$ of symbols $\omega_n \in \{\mathrm{u}, \mathrm{d}\}$ for $n = 1, \ldots, N$. There are 2^N elements in this Ω.

The objective market probability P on Ω is such that

$$P(\omega) = p^n(1-p)^{N-n},$$

for any sequence $\omega \in \Omega$, where n is the number of u's appearing in the sequence. The random variables $K(n)$ for $n = 1, \ldots, N$ can be defined by

$$K(n,\omega) = \begin{cases} u & \text{if } \omega_n = \mathrm{u}, \\ d & \text{if } \omega_n = \mathrm{d}. \end{cases}$$

The risk neutral probability can be written as

$$P^*(\omega) = p^{*n}(1-p^*)^{N-n},$$

and for the stock price we have

$$S(n,\omega) = S(0)(1+u)^n(1+d)^{N-n},$$

for $n = 0, 1, \ldots, N$, where n is the number of u's appearing in the sequence.$\omega \in \Omega$. We shall often write $S^{\omega}(n)$ instead of $S(n, \omega)$.

This multi period binary tree model of a financial market is sometimes called the *Cox-Ross-Rubinstein* (CRR) model. It has three parameters r, u, d, which need to be matched to market data. This matching procedure is called *calibration.*

2.1 European Options and the CRR Formula

Consider a European option with exercise time N and payoff $h(S(N))$.

The term 'European' means that the option can be exercised only at time N, when the holder is entitled to receive the payoff $h(S(N))$. (There is also another class of options which can be exercised at any time up to and including time N. They are called American options and will be briefly discussed later.)

The option price $D(n)$ for $n = N, \ldots, 0$ can then be computed by backward induction as follows:

- At time N put
$$D(N) = h(S(N)). \tag{2.1}$$
- If for some $n = 1, \ldots, N$ we have already computed $D(n)$ has already each node at time n, then consider the single period subtree
$$\begin{tikzcd} & S^u(n) = (1+u)S(n-1) \\ S(n-1) \arrow[ur] \arrow[dr] & \\ & S^d(n) = (1+d)S(n-1) \end{tikzcd}$$
and put
$$D(n-1) = \frac{1}{1+r}\left[p^* D^u(n) + (1-p^*)D^d(n)\right]. \tag{2.2}$$

Proceeding in this way, we shall eventually reach the option price $D(0)$.

Example 13 In a two period binary tree model, i.e. a model with $N = 2$, and with initial stock price $S(0) = 100$ and parameters

$$r = 0.1, \quad u = 0.2, \quad d = -0.1,$$

consider a European call option with strike price $K = 100$ and exercise time $N = 2$. To price the option, we first compute the risk neutral probability

$$p^* = \frac{r-d}{u-d} = \frac{0.1-(-0.1)}{0.2-(-0.1)} = \frac{2}{3}.$$

We then build the tree of stock prices, and list values for the payoff of the European option:

$$\begin{tikzcd}
& & \begin{array}{l} S^{uu}(2) = 144 \\ h(S^{uu}(2)) = 44 \end{array} \\
& S^{u}(1) = 120 \arrow[ur] \arrow[dr] & \\
S(0) = 100 \arrow[ur] \arrow[dr] & & \begin{array}{l} S^{ud}(2) = S^{du}(2) = 108 \\ h(S^{ud}(2)) = h(S^{du}(2)) = 8 \end{array} \\
& S^{d}(1) = 90 \arrow[ur] \arrow[dr] & \\
& & \begin{array}{l} S^{dd}(2) = 81 \\ h(S^{dd}(2)) = 0 \end{array}
\end{tikzcd}$$

Next, we can construct the tree of option prices using (2.1) and (2.2):

$$\begin{tikzcd}
& & D^{uu}(2) = 44.00 \\
& D^{u}(1) \approx 29.09 \arrow[ur] \arrow[dr] & \\
D(0) \approx 19.10 \arrow[ur] \arrow[dr] & & D^{ud}(2) = D^{du}(2) = 8.00 \\
& D^{d}(1) \approx 4.85 \arrow[ur] \arrow[dr] & \\
& & D^{dd}(2) = 0.00
\end{tikzcd}$$

Proposition 14 *The option price $D(0)$ is the expectation under the risk neutral probability of the discounted payoff, namely*

$$D(0) = \frac{1}{(1+r)^N} \mathbb{E}^*(h(S(N))). \tag{2.3}$$

Proof outline We know that

$$D(N) = h(S(N)).$$

We also know how to find the option value $D(N-1)$ at time $N-1$, since this involves just a single period binary model like this:

$$\begin{tikzcd}
& S^{u}(N) = (1+u)S(N-1) \\
S(N-1) \arrow[ur] \arrow[dr] & \\
& S^{d}(N) = (1+d)S(N-1)
\end{tikzcd}$$

We have, therefore

$$\begin{aligned}
D(N-1) &= \frac{1}{1+r} \left[p^* D^u(N) + (1-p^*)\, D^d(N) \right] \\
&= \frac{1}{1+r} \left[p^* h((1+u)S(N-1)) + (1-p^*)\, h((1+d)S(N-1)) \right].
\end{aligned}$$

Next we take one more step back to determine $D(N-2)$. The tree fragment looks like this:

$$\begin{tikzcd}
 & S^u(N-1) = (1+u)S(N-2) \\
S(N-2) \arrow[ur] \arrow[dr] & \\
 & S^d(N-1) = (1+d)S(N-2)
\end{tikzcd}$$

We have

$$\begin{aligned}
D(N-2) &= \frac{1}{1+r}\left[p^* D^u(N-1) + (1-p^*)\, D^d(N-1)\right] \\
&= \frac{1}{(1+r)^2}[p^*\left(p^* h((1+u)^2 S(N-2)) + (1-p^*)\, h((1+u)(1+d)S(N-2))\right) \\
&\qquad + (1-p^*)\left(h((1+u)(1+d)S(N-2)) + (1+d)^2 S(N-2))\right)] \\
&= \frac{1}{(1+r)^2}[p^* h((1+u)^2 S(N-2)) + 2p^*(1-p^*)h((1+u)(1+d)S(N-2) \\
&\qquad + (1-p^*)\, h((1+d)^2 S(N-2)]
\end{aligned}$$

Proceeding this way by backward induction, we can find eventually that

$$\begin{aligned}
D(0) &= \frac{1}{(1+r)^N}\sum_{n=0}^{N}\frac{N!}{n!(N-n)!}p^{*n}(1-p^*)^{N-n}h((1+u)^n(1+d)^{N-n}S(0)) \\
&= \frac{1}{(1+r)^N}\mathbb{E}^*(h(S(N))).
\end{aligned}$$

■

Example 15 (Example 13 continued) We can compute the option price directly form (2.3):

$$\begin{aligned}
D(0) &= \frac{1}{(1+r)^N}\mathbb{E}^*(h(S(2)) \\
&= \frac{1}{(1.1)^2}\left[\left(\frac{2}{3}\right)^2 \times 44 + 2 \times \frac{2}{3} \times \frac{1}{3} \times 8 + \left(\frac{1}{3}\right)^2 \times 0\right] \\
&\approx 19.10.
\end{aligned}$$

Corollary 16 *In particular, for a call option with strike price K and exercise time N we obtain the famous Cox-Ross-Rubinstein (CRR) formula*

$$\begin{aligned}
C(0) &= \frac{1}{(1+r)^N}\sum_{n=0}^{N}\frac{N!}{n!(N-n)!}p^{*n}(1-p^*)^{N-n}\max(0, S(0)(1+u)^n(1+d)^{N-n} - K) \\
&= \frac{1}{(1+r)^N}\sum_{n=n_0}^{N}\frac{N!}{n!(N-n)!}p^{*n}(1-p^*)^{N-n}(S(0)(1+u)^n(1+d)^{N-n} - K),
\end{aligned}$$

where summation starts with the smallest number n_0 such that

$$S(0)(1+u)^{n_0}(1+d)^{N-n_0} > K.$$

2.1.1 Replication in the Multi Period Binary Model

In this case, when a replicating portfolio is created at time 0, it is then possible to rebalance the positions in a portfolio at each intermediate trading date between times 0 and N. As a result, a sequence of portfolios (x_n, y_n) for $n = 1, \ldots, N$ should be considered, where (x_n, y_n) will be understood as the portfolio of cash and stock held during the time period between trading dates number $n-1$ an n (that is, between times $(n-1)\delta$ and $n\delta$). Such a sequence of portfolios is called a trading strategy.

There are two conditions that a trading strategy must satisfy:

- For any $n = 1, \ldots, N$, when creating portfolio (x_n, y_n) at time $n-1$, only the knowledge of stock prices $S(0), \ldots, S(n-1)$ up to and including that time can be used. This is because at time $n-1$ we cannot predict the values of the future stock prices $S(n), \ldots, S(N)$. Mathematically, (x_n, y_n) should be a function of $S(0), \ldots, S(n-1)$ but not of $S(n), \ldots, S(N)$. A sequence of portfolios (x_n, y_n) satisfying this property is called predictable.
- For any $n = 1, \ldots, N-1$, portfolio (x_{n+1}, y_{n+1}) must be created by rebalancing the portfolio (x_n, y_n) without injection or withdrawal of any funds. This means that

  $$(1+r)x_n + S(n)y_n = x_{n+1} + S(n)y_{n+1}.$$

  Such a trading strategy is called self financing.

The value of a trading strategy at time $n = 0, 1, \ldots, N$ will be denoted by

$$V(n) = \begin{cases} x_1 + S(0)y_1 & \text{for } n = 0, \\ (1+r)x_n + S(n)y_n & \text{for } n = 1, \ldots, N. \end{cases}$$

For a self financing trading strategy the value can also be expressed as

$$V(n) = \begin{cases} x_{n+1} + S(n)y_{n+1} & \text{for } n = 0, \ldots, N-1, \\ (1+r)x_N + S(N)y_N & \text{for } n = N. \end{cases}$$

Definition 17 *By a replicating strategy for a European option with payoff $h(S(N))$ and exercise time N we mean a predictable self financing trading strategy (x_n, y_n) for $n = 1, \ldots, n$ whose time N value matches the option payoff,*

$$V(N) = h(S(N)).$$

Example 18 (Example 13 continued)

$$\begin{tikzcd}
 & & D^{uu}(2) = 44.00 \\
 & D^u(1) \approx 29.09 \arrow[ur, "{x_2^u = -82.645,\ y_2^u = 1.00}"] \arrow[dr] & \\
D(0) \approx 19.10 \arrow[ur, "{x_1 = -61.7,\ y_1 = 0.808}"] \arrow[dr] & & D^{ud}(2) = D^{du}(2) = 8.00 \\
 & D^d(1) \approx 4.85 \arrow[ur, "{x_2^d = -19.835,\ y_2^d = 0.29630}"] \arrow[dr] & \\
 & & D^{dd}(2) = 0.00
\end{tikzcd}$$

2.2 American Options

An American option differs form a European one in that the holder has the right to exercise it at any time step $n = 0, 1, \ldots, N$, rather than just at the expiry time T. We shall denote by $Y(n) = h(S(n))$ the payoff to be received at time step $n = 0, 1, \ldots, N$ if the holder decides to exercise at that time.

For example, exercising an American put with strike price K at time step $n = 0, 1, \ldots, N$ enables the holder to buy a share of stock for K rather than for $S(n)$. This amounts to receiving the payoff $Y(n) = \max(K - S(n), 0)$.

The price of the American option at time step $n = 0, 1, \ldots, N$ will be denoted by $Z(n)$.

In the binomial tree model we can price an American option by backward induction, extending the method discussed for European options. At expiry time N, if the option has not yet been exercised, then there is no longer any choice left as to when it can be exercised. As a result, the value of the option equals the payoff,

$$Z(N) = Y(N).$$

Now suppose that for some $n = 0, 1, ..., N - 1$ we have already figured out the option price $Z(n + 1)$ at time step $n + 1$. How can we find $Z(t)$? We take any node $\omega_1 \ldots \omega_n$ at time n and consider the single period tree fragment

$$\begin{tikzcd}
 & S^u(n+1) = (1+u)S(n) \\
S(n) \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\
 & S^d(n+1) = (1+d)S(n)
\end{tikzcd}$$

As for European options, we compute the expectation

$$V(n) = \frac{1}{1+r} \left[p^* Z^u(n+1) + (1 - p^*) Z^d(n+1) \right].$$

This is how much the option would be worth if the holder were unable to exercise at time step n. It is called the continuation value. However, the holder does have the right to exercise at time step n and to receive the payoff $Y(n)$. If $Y(n) < V(n)$, the holder would not want to exercise the option at time step n. But if $Y(n) \geq V(n)$, then it makes sense to exercise. In effect, at time step n the holder is able to choose between the higher of $Y(n)$ and $V(n)$ by either exercising or not, and therefore the value of the option must be

$$Z(n) = \max(Y(n), V(n)).$$

This can be proved precisely by an arbitrage argument, but this is left as an exercise for the reader.

Example 19 We consider the binary model with $N = 2$ time steps, initial stock price $S(0) = 100$ and parameters

$$r = 0.1, \quad u = 0.2, \quad d = -0.2.$$

and an American put option with strike price 103 which expires at time step 2. To price the option, we first compute the risk neutral probability

$$p^* = \frac{r-d}{u-d} = \frac{0.1-(-0.2)}{0.2-(-0.2)} = \frac{3}{4}.$$

The stock prices and payoffs can be represented by the tree

$$\begin{tikzcd}
 & & \begin{array}{l} S^{uu}(2) = 144 \\ Y^{uu}(2) = 0 \end{array} \\
 & \begin{array}{l} S^{u}(1) = 120 \\ Y^{u}(1) = 0 \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{l} S(0) = 100 \\ Y(0) = 3 \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{l} S^{ud}(2) = S^{du}(2) = 96 \\ Y^{ud}(2) = Y^{du}(2) = 7 \end{array} \\
 & \begin{array}{l} S^{d}(1) = 80 \\ Y^{d}(1) = 23 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{l} S^{dd}(2) = 64 \\ Y^{dd}(2) = 39 \end{array}
\end{tikzcd}$$

Next, we construct the tree listing the continuation values $V(n)$ and option prices $Z(n)$

$$\begin{tikzcd}
 & & \begin{array}{l} V^{uu}(2) = 0 \\ Z^{uu}(2) = 0 \end{array} \\
 & \begin{array}{l} V^{u}(1) = 1.5909 \\ Z^{u}(1) = 1.5909 \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{l} V(0) = 6.3120 \\ Z(0) = 6.3120 \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{l} V^{ud}(2) = V^{du}(2) = 7 \\ Z^{ud}(2) = Z^{du}(2) = 7 \end{array} \\
 & \begin{array}{l} V^{d}(1) = 13.636 \\ Z^{d}(1) = 23 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{l} V^{dd}(2) = 39 \\ Z^{dd}(2) = 39 \end{array}
\end{tikzcd}$$

It only makes sense to exercise an American option when $Y(n) > 0$ and $Z(n) = Y(n)$. There is no point exercising the option when $Y(n) = 0$ and it should not be exercises when $Z(n) > Y(n)$ because in such case it would be better to sell it for $Z(n)$ rather than to receive the payoff $Y(n)$ if the option is exercised. We call the random variable

$$\tau = \begin{cases} \min\{n = 0, \ldots, N | Z(n) = Y(n) > 0\} & \text{if this set is non-empty} \\ \infty & \text{if this set is empty} \end{cases}$$

the early exercise time (sometimes called the early exercise boundary). The value ∞ means that the option will never be exercised.

Example 20 (Exercise 19 continued) In this example the early exercise time is

$$\tau(\omega) = \begin{cases} \infty & \text{if } \omega = uu, \\ 2 & \text{if } \omega = ud, \\ 1 & \text{if } \omega = du, \\ 1 & \text{if } \omega = dd. \end{cases}$$