Mathematical Finance

Introduction to Binary Tree Models,
Stochastic Calculus and Black-Scholes Theory

Lecture Notes

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1 Single Period Binary Model

Example 1 First a simple but important example. Consider a time period between the present time $t = 0$ and some fixed future time instant $T > 0$, say half a year. We shall be measuring time in years, so half a year means that $T = \frac{1}{2}$. Over this time period we shall consider investment in the following financial securities, or portfolios consisting of such securities:

- Cash (or bonds)
  - This is considered a risk free security in the sense that it can be invested at a given rate of return $r \geq 0$, say $r = 5\%$. It means that an investment of, say \$100 made at time 0 will become \$105 at time T, as illustrated by the diagram

    $$\$100 \longrightarrow \$105$$

  - Small investors can achieve this simply paying cash into a bank account, where it will earn interest. Large institutions or governments can achieve the same effect by trading in bonds.
  - The same rate of return r is assumed for deposits and loans. This is almost true for large institutions, but not for small investors.

- Stock
  - A risky security. Its price today is known, say \$100 per share, but the future price at time T is unknown, and can go up or down.
  - We need a good mathematical model for stock prices.
  - For now we consider a very simple model according to which the stock price at time T can go up to \$125 or down to \$95 with fifty-fifty probability, as in the diagram

    $$\$100 \begin{array}{c} \stackrel{p=0.5}{\nearrow} \\ \underset{1-p=0.5}{\searrow} \end{array} \begin{array}{c} \$125 \\ \\ \$95 \end{array}$$

  - This is an example of a single period binomial model. It is simplistic, but will help us to introduce and understand some important new concepts in an easy context. Then we shall hopefully be ready to move on to some more realistic but also more complicated stock price models.
  - We assume that an investor can have a positive or a negative position in stock, called long and short positions, respectively. A short position can be achieved by borrowing some stock and selling it. The stock will eventually have to be bought back and returned to the owner. This is a good assumption for large financial institutions but not for small investors.

- For mathematical convenience we assume that fractional parts of a share of stock can be traded. Once again, this is an acceptable assumption for large institutions, who hold and trade large portfolios of shares, but not so for small investors.

- Call option

  - This is a contract which gives the holder the right but not an obligation to buy a share of the stock at a specified time T, here $T = \frac{1}{2}$ (half a year), for a price agreed in advance, say $K = \$105$.
  - We say that the call option can be exercised at time T at strike price K.
  - A call option can be viewed as insurance against the stock price going up above the strike price.
  - The option value at time T, called the payoff, is easy to figure out. It depends on the stock price at that time. If the stock price happens to go up to \$125 at time T, then it would be profitable to exercise the option, buying the stock for \$105. Since the stock can be sold immediately for \$125, this will bring a profit of \$20. On the other hand, if the stock price turns out to have fallen down to \$95 at time T, then it would make little sense to exercise the option and buy stock for \$105. In this case the option will be worthless. This is shown in the diagram

    $$? \quad \begin{array}{l} \overset{p=0.5}{\nearrow} \quad \$20 \\ \underset{1-p=0.5}{\searrow} \quad \$0 \end{array}$$

    The probabilities are the same as those for the stock price going up/down.
  - We need to figure how much the option is worth at time 0. This is an important question for an investor who wants to buy or sell the option.

1.1 Call Option Price

Example 2 (Example 1 continued) We shall try to figure out the call option price at time 0.

- A common guess is that because the option pays \$20 or \$0 with probability fifty-fifty, it should be worth

  $$0.5 \times \$20 + 0.5 \times \$0 = \$10$$

  at time 0. Unfortunately, this guess is wrong, and it is important to understand why.

In the spreadsheet `CallExample.xls` we can see that selling 3 calls for $3 \times \$10 = \$30$, borrowing \$170 in cash at 5% and buying 2 shares for $2 \times \$100 = \$200$ will cost nothing because the balance of these three transactions is nil at time 0. However, no matter whether the stock will go up or down, this portfolio will be worth \$11.50 at time T.

This kind of profit, made out of nothing and without any risk of loss is called arbitrage. If such a strategy existed in the real world, everyone (or at least everyone who knows how to do it) would like to follow it. While at it, why not to multiply all positions by, say $1,000,000$, so the final profit would be $\$11,500,000$ rather than the meager \$11.50, whereas the initial financial outlay would still be \$0 and no risk would be involved.

The guess that the option price should be \$10 is wrong because it leads to an arbitrage opportunity. An opportunity of this kind is too good to be true and therefore it is safe to assume that it cannot happen.

- A more sophisticated argument is that the \$20 to be received in half a year is worth less in today's money, and should be discounted to $\frac{\$20}{1.05} \approx \$19.05$ using the 5% rate, and then the time 0 option price should be figured out using the probabilities of up/down price movements:

  $$0.5 \times \$19.05 + 0.5 \times \$0 \approx \$9.52$$

  However, we can see from the spreadsheet that this would also lead to arbitrage. This time one would need to sell 3 calls, borrow \$171.43 in cash, and buy 2 shares of stock. The initial balance of these transactions would be 0 and the final value \$10, no matter whether the stock goes up or down. This is arbitrage opportunity, so the option price of \$9.25 must also be wrong.

- Many other trials lead to similar arbitrage opportunities, but it is possible by trial an error to home in on an option price of about \$6.35 for which the arbitrage opportunity seems to disappear.

Is \$6.35 the correct option price, then? We shall need to work our apprenticeship to see what kind of magic may be at work behind the scenes.

1.2 Replicating the Option

Example 3 (Example 1 continued) At time 0 we set up a portfolio (x, y) consisting of a cash amount x and y shares of stock so that the value of the portfolio at time T equals that of the option no matter whether stock goes up or down:

$$\begin{aligned} 1.05x + 125y &= 20, \\ 1.05x + \ \ 95y &= \ \ 0. \end{aligned}$$

The solution to this system of equations is

$$x \approx -60.3174, \quad y \approx 0.666666.$$

This procedure is called a replicating the option, and (x, y) is called the replicating portfolio.

The value of the replicating portfolio at time 0 is

$$x + 100y \approx -60.3174 + 100 \times 0.666666 \approx 6.3492.$$

It is the same value (up to a rounding error) we have homed at in the spreadsheet by trying to avoid arbitrage.

This example suggests a deeper connection, which can be stated informally as follows:

Conjecture 4 *The time* 0 *value of the replicating portfolio should be equal to that of the option, or else an arbitrage opportunity would exist.*

By an arbitrage opportunity we mean, informally, that it would be possible to follow a strategy with nil initial cost, no risk of financial loss, and positive income at least in some future scenarios. In short, it would be possible to make some money out of nothing, and without taking any risk whatsoever.

1.3 Replication: General Properties

We shall now present some formulae and properties for replicating portfolios in order to make the informal arguments more general and precise.

We still work in a single period binary model, but instead of concrete numbers, let us use general symbols. Namely, $r \geq 0$ will be the rate of return of the risk free security (cash) over the period from time 0 (today) to some fixed future date $T > 0$. Let $S(0)$ be today's stock price, which is obviously known, and let $S(T)$ denote future stock price at time T. The stock is a risky security, so $S(T)$ is unknown today. Mathematically, this means that $S(T)$ is a random variable.

In the single period binary model we assume that the random variable $S(T)$ can take two values, $S^u(T)$ with probability p, and $S^d(T)$ with probability $1-p$, where $0 < p < 1$ and $S^u(T) > S^d(T)$. The superscripts u, d stand for 'up' and 'down' stock price movements. This can be represented diagrammatically as

$$S(0) \quad \begin{matrix} \overset{p}{\nearrow} & S^u(T) \\ \underset{1-p}{\searrow} & S^d(T) \end{matrix}$$

In Example 1 we considered a call option. Apart from calls, there are various other options actively traded in the markets.

Example 5 Here are some examples of option payoffs. In all these examples K, L are given values such that $K < L$.

- For a call with exercise time T and strike price K the payoff (that is, the value of the option at T) can be written as $\max(S(T) - K, 0)$;

- For a put option, which gives the holder the right but not an obligation to sell the stock for K at time T, the payoff can be written as $\max(K - S(T), 0)$.
- For a digital call with strike price K and exercise time T the payoff is 1 if $S(T) \geq K$ and 0 if $S(T) < K$.
- For a digital put with strike price K and exercise time T the payoff is 0 if $S(T) \geq K$ and 1 if $S(T) < K$.
- For a straddle struck at K with exercise time T the payoff is $\max(K - S(T), S(T) - K)$.
- For a strangle the payoff is $\max(K - S(T), 0, S(T) - L)$.
- For a bull spread the payoff is $\max(0, S(T) - K, L - K)$.
- For a bear spread the payoff is $\max(L - K, L - S(T), 0)$.
- For a butterfly the payoff is $\max(0, \min(S(T) - K, L - S(T))$.
- For a double digital option the payoff is 1 is $K \leq S(T) \leq L$ and 0 otherwise.

All such options share a common feature: the payoff is a function $h(S(T))$ of the stock price $S(T)$.

Options are often referred to as derivative securities because their value derived from that of the stock. Another name that is used for options is contingent claims because their value is contingent on (that is, depends on) the stock price.

We shall denote by $D(0)$ the time 0 value of a derivative security with payoff function $h(S(T))$, and by $D(T)$ the value at time T. Obviously, the value at time T is just the payoff

$$D(T) = h(S(T)).$$

The question is, how to compute $D(0)$ so it does not to give rise to an arbitrage opportunity.

This calls for a precise definition of arbitrage. First, we consider a portfolio (x, y, z) consisting of an amount x in cash, y shares of stock and z options. The time 0 value of this portfolio is

$$V(0) = x + S(0)y + D(0)z.$$

The value at time T will be

$$V(T) = (1 + r)x + S(T)y + D(T)z.$$

Definition 6 An arbitrage opportunity is a portfolio with initial value $V(0) = 0$ and final value $V(T) \geq 0$ such that

$$V(T) > 0 \quad \text{with positive probability.}$$

Next we shall derive formulae the replicating portfolio (x, y) of cash and stock for a derivative security with general payoff $h(S(T))$. The final value of the replicating portfolio should be equal to the payoff, so

$$(1+r)x + S(T)y = h(S(T)).$$

This is an equality between random variables. It should be satisfied irrespective of the actual value of $S(T)$ that will be realised at time T. In the single period binary model this gives a system of two equations

$$\begin{aligned}(1+r)x + S^u(T)y &= h(S^u(T)),\\ (1+r)x + S^d(T)y &= h(S^d(T)).\end{aligned}$$

It is not hard to write down the solution:

$$y = \frac{h(S^u(T)) - h(S^d(T))}{S^u(T) - S^d(T)}, \tag{1}$$

$$x = \frac{1}{1+r} \frac{h(S^d(T))S^u(T) - h(S^u(T))S^d(T)}{S^u(T) - S^d(T)}. \tag{2}$$

Proposition 7 *If there is no arbitrage opportunity, then the value of the derivative security at time* 0 *must be equal to that of the replicating portfolio,*

$$D(0) = x + S(0)y,$$

where x, y *are given by* (1), (2).

Proof We shall prove this result by contradiction. First suppose that

$$D(0) > x + S(0)y.$$

In that case we could follow this strategy:

- At time 0 sell one option for $D(0)$, buy y shares of stock for $S(0)y$, and invest the cash balance $D(0) - S(0)y$ at the rate r, that is, set up a portfolio $(D(0) - S(0)y, y, -1)$ with initial value $V(0) = 0$.
- At time T the stock will be worth $S(T)y$, the cash investment will grow to $(1+r)(D(0) - S(0)y)$, and the option value will be $h(S(T))$. The value of the portfolio $(D(0) - S(0)y, y, -1)$ will therefore be

$$\begin{aligned}V(T) &= (1+r)(D(0) - S(0)y) + S(T)y - h(S(T))\\ &> (1+r)x + S(T)y - h(S(T)) = 0.\end{aligned}$$

This would, therefore, an arbitrage opportunity. On the other hand, if

$$D(0) < x + S(0)y$$

then by a similar argument the portfolio $(-D(0) + S(0)y, -y, 1)$ would be an arbitrage opportunity. If there are no arbitrage opportunities, then the only possibility left is that $D(0) = x + S(0)y$, as claimed. ■

Example 8 (Example 1 continued) The portfolios constructed in the proof of Proposition 7 are those computed in the spreadsheet accompanying Exercise 1 (except for a constant factor chosen so as to have the stock and option positions expressed as whole numbers).

Remark 9 The derivative security price $D(0)$ does not depend on p, the actual probability (often called the objective probability or market probability) of up/down stock price movements. This is a remarkable fact, which may well appear paradoxical on first sight. Option pricing relies on a link between the option payoff and the underlying stock price rather then on the actual probability of stock price movements.

1.4 Risk Neutral Probability

Proposition 7 provides the derivative security price as $D(0) = x + S(0)y$, where x, y are given by (1), (2). Substituting these expressions for x and y, we get

$$\begin{aligned} D(0) &= x + S(0)y \\ &= \frac{1}{1+r}\frac{h(S^d(T))S^u(T) - h(S^u(T))S^d(T)}{S^u(T) - S^d(T)} + S(0)\frac{h(S^u(T)) - h(S^d(T))}{S^u(T) - S^d(T)} \\ &= \frac{(1+r)S(0) - S^d(T)}{S^u(T) - S^d(T)}\frac{h(S^u(T))}{1+r} + \frac{S^u(T) - (1+r)S(0)}{S^u(T) - S^d(T)}\frac{h(S^d(T))}{1+r} \\ &= p^*\frac{h(S^u(T))}{1+r} + (1-p^*)\frac{h(S^d(T))}{1+r}, \end{aligned}$$

where

$$p^* = \frac{(1+r)S(0) - S^d(T)}{S^u(T) - S^d(T)}. \tag{3}$$

Observe that if

$$S^d(T) < (1+r)S(0) < S^u(T), \tag{4}$$

then

$$0 < p^* < 1,$$

can be interpreted as a probability. It is called the risk neutral probability. The above expression for the derivative security price can then be written as

$$D(0) = p^*\frac{h(S^u(T))}{1+r} + (1-p^*)\frac{h(S^d(T))}{1+r} = \mathbb{E}^*(\frac{h(S(T))}{1+r})$$

in terms of the expectation of the discounted payoff $\frac{h(S(T))}{1+r}$ under the risk neutral probability.

Proposition 10 *If there is no arbitrage opportunity, then the value of the derivative security at time* 0 *is equal to*

$$D(0) = \mathbb{E}^*(\frac{h(S(T))}{1+r}),$$

where expectation is taken under the risk neutral probability given by (3).

Proof All that remains to be proved is that $S^d(T) < (1+r)S(0) < S^u(T)$.

First suppose that in fact $S^d(T) \geq (1+r)S(0)$. Then we could follow this strategy:

- At time 0 borrow the amount $S(0)$ in cash at the rate r and use it to buy one share of stock for $S(0)$, setting up a portfolio $(-S(0), 1)$ of cash and stock with initial value $V(0) = 0$.
- At time T sell the stock for $S(T)$ and pay $(1+r)S(0)$ to clear the cash loan with interest.
  - If stock goes down, the final value of the portfolio $(-S(0), 1)$ will be

    $$V^d(T) = S^d(T) - (1+r)S(0) \geq 0.$$

  - If stock goes up, the final value of the portfolio $(-S(0), 1)$ will be

    $$\begin{aligned} V^u(T) &= S^u(T) - (1+r)S(0) \\ &> S^d(T) - (1+r)S(0) \geq 0. \end{aligned}$$

  In either case $V(T) \geq 0$, and we would have $V(T) > 0$ with probability $1 - p > 0$, which means that an arbitrage opportunity would exist.

Next suppose that $(1+r)S(0) \geq S^u(T)$. Then

- At time 0 sell short one share of stock for $S(0)$ and invest the cash at rate r, that is, construct the portfolio $(S(0), -1)$ of cash and stock with initial value $V(0) = 0$.
- At time T withdraw the cash investment, which will be worth $(1+r)S(0)$ complete with interest, and pay $S(T)$ to purchase a share of stock in order to close the short position in stock.
  - If stock goes down, the final value of the portfolio $(S(0), -1)$ will be

    $$\begin{aligned} V^d(T) &= (1+r)S(0) - S^d(T) \\ &> (1+r)S(0) - S^u(T) \geq 0. \end{aligned}$$

  - If stock goes up, the final value of the portfolio $(S(0), -1)$ will be

    $$V^u(T) = (1+r)S(0) - S^u(T) \geq 0.$$

  This means that an arbitrage strategy would exist in this case too.

Because it is assumed that no arbitrage strategies can exist, the only possibility left is that $S^d(T) < (1+r)S(0) < S^u(T)$. ■

Example 11 (Example 1 continued) The risk neutral probability in this example is

$$p^* = \frac{105 - 95}{125 - 95} = \frac{1}{3}.$$

The call option price can therefore be computed as

$$\frac{1}{3}\frac{20}{1.05} + \frac{2}{3}\frac{0}{1.05} \approx 6.3492.$$

Remark 12 We have, after all, been able to express the derivative security price as an expectation of the discounted payoff. However, the expectation is not computed using the actual market probability p of up/down stock price movements. Instead, we use the risk neutral probability p^*, an artificial mathematical object. Nevertheless, it is a convenient mathematical object, remarkably simplifying the computation of the option price.