

# Mathematical Finance

Introduction to Binary Tree Models,  
Stochastic Calculus and Black-Scholes Theory

## Exercises

Tomasz Zastawniak  
Department of Mathematics  
University of York  
tz506@york.ac.uk

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1. Using the data for the single period binary model in Example 1
  - (a) compute the replicating portfolio for a put option with strike price  $K = \$105$  and exercise time  $T = \frac{1}{2}$ ;
  - (b) compute the time 0 put price using the replicating portfolio;
  - (c) compute the time 0 put price using the risk neutral probability;
  - (d) show how to achieve arbitrage if the time 0 put price were \$5.
2. Prove the *put-call parity* relationship

$$C(0) = P(0) + S(0) - \frac{K}{1+r}$$

between the prices of a call and put option on the same underlying stock, with the same exercise price  $K$  and expiry time  $T$ , where  $r \geq 0$  is the rate of return on a risk free investment between times 0 and  $T$ .

*Hint:* If the equality were violated, try to construct an arbitrage portfolio consisting of four securities: call and put options, cash and stock.

3. Consider a five period binary model with the following parameters

$$S(0) = 100, \quad u = 0.04, \quad d = -0.02, \quad r = 0.02.$$

- (a) Using a spreadsheet program of your choice compute the price of a European call and a European put option with a strike price  $X = 98$  and expiry date  $T = 5$ .
- (b) Using a spreadsheet program of your choice compute the price of an American call and an American put option with a strike price  $X = 98$  and expiry date  $T = 5$ . Determine also at which moment the buyer should exercise these options.
4. Let  $C_E(0)$  and  $C_A(0)$  denote the prices of a European call and an American call, respectively. Assume that both options have the same strike price and expiry date. Using an arbitrage argument, show that the prices of a European call and an American call option are the same

$$C_E(0) = C_A(0).$$

5. Show that

$$\mathbb{E}[B(s)B(t)] = \min(s, t).$$

6. Let  $c > 0$ . Show that if  $B(t)$  is a Brownian motion, then

$$W(t) = \frac{1}{c}B(ct)$$

is also a Brownian motion.

7. Show that

$$\lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} (B(t_{i+1}) - B(t_i))^2 = T,$$

where the limit is taken in  $L^2$ .

8. Use the Itô formula to show that

$$X(t) = Ce^{aB(t) - \frac{1}{2}a^2t}$$

is a solution to the stochastic differential equation

$$dX(t) = aX(t)dB(t).$$