Mathematical Finance

Introduction to Binary Tree Models, Stochastic Calculus and Black-Scholes Theory

Exercises

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1. Using the data for the single period binary model in Example 1

   (a) compute the replicating portfolio for a put option with strike price $K = \$105$ and exercise time $T = \frac{1}{2}$;

   (b) compute the time 0 put price using the replicating portfolio;

   (c) compute the time 0 put price using the risk neutral probability;

   (d) show how to achieve arbitrage if the time 0 put price were \$5.

2. Prove the *put-call parity* relationship

$$C(0) = P(0) + S(0) - \frac{K}{1+r}$$

   between the prices of a call and put option on the same underlying stock, with the same exercise price K and expiry time T, where $r \geq 0$ is the rate of return on a risk free investment between times 0 and T.

   Hint: If the equality were violated, try to construct an arbitrage portfolio consisting of four securities: call and put options, cash and stock.

3. Consider a five period binary model with the following parameters

$$S(0) = 100, \quad u = 0.04, \quad d = -0.02, \quad r = 0.02.$$

(a) Using a spreadsheet program of your choice compute the price of a European call and a European put option with a strike price $X = 98$ and expiry date $T = 5$.

(b) Using a spreadsheet program of your choice compute the price of an American call and an American put option with a strike price $X = 98$ and expiry date $T = 5$. Determine also at which moment the buyer should exercise these options.

4. Let $C_E(0)$ and $C_A(0)$ denote the prices of a European call and an American call, respectively. Assume that both options have the same strike price and expiry date. Using an arbitrage argument, show that the prices of a European call and an American call option are the same

$$C_E(0) = C_A(0).$$

5. Show that

$$\mathbb{E}\left[B(s)B(t)\right] = \min(s,t).$$

6. Let $c > 0$. Show that if $B(t)$ is a Brownian motion, then

$$W(t) = \frac{1}{c}B(c^2 t)$$

is also a Brownian motion.

7. Show that

$$\lim_{N\to\infty} \sum_{i=1}^{N-1} \left(B(t_{i+1}) - B(t_i)\right)^2 = T,$$

where the limit is taken in L^2.

8. Use the Itô formula to show that

$$X(t) = Ce^{aB(t)-\frac{1}{2}a^2 t}$$

is a solution to the stochastic differential equation

$$dX(t) = aX(t)dB(t).$$

9. Show that if $u(t,x)$ is a solution to the heat equation

$$u'_t = \frac{1}{2}u''_{xx}$$

with initial condition

$$u(0,x) = \varphi(x),$$

then the Feynman-Kac formula holds:

$$u(t,x) = \mathbb{E}\left[\varphi(x + B(t))\right].$$

Hint: Use the Itô formula for $f(s,B(s)) = u(t-s, x+B(s))$.

10. Show that

$$\mathbb{E}\left[e^{\int_0^T t dB(t)}\right] = e^{\frac{1}{6}T^3}.$$

Hint: Apply the Itô formula to compute $d\left[e^{\int_0^t s dB(s) - \frac{1}{6}t^3}\right]$.