# Mathematical Finance

## Introduction to Binary Tree Models, Stochastic Calculus and Black-Scholes Theory

### Assignment 3

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1. Denote by $P(0)$ the price of a European put option with exercise time $T$ and strike price $K$ in the Black-Scholes model. Starting from the formula

$$P(0) = \mathbb{E}^* \left[ e^{-rT} h(S(T)) \right],$$

where

$$h(S(T)) = \max(K - S(T), 0)$$

is the put payoff, and following a similar argument as for a call option presented in the lectures, derive the following formula for the put option price:

$$P(0) = e^{-rT} K N(-d_2) - S(0) N(-d_1),$$

where

$$d_1 = \frac{\ln \frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln \frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}},$$

and where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx$$

is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

2. A digital call option with strike price $K$ exercised at time $T$ has payoff

$$h(S(T)) = \begin{cases} 1 & \text{if } S(T) \geq K, \\ 0 & \text{if } S(T) < K. \end{cases}$$

We denote the time 0 price of such a European digital call option by $D(0)$. Starting with the formula

$$D(0) = \mathbb{E}^* \left[ e^{-rT} h(S(T)) \right]$$

and working in the Black-Scholes model, derive the following expression for the price of a digital call option:

$$D(0) = e^{-rT} N(d_2),$$

where $d_2$ and $N$ are as above.