Real Analysis - Assignment II
Submission deadline: Thursday (evening), the 10th of December.

1. Let $F : [0,1] \to [0,1]$ be defined as follows:

   - $F(r) = 1/q$ for $r = \frac{p}{q} > 0$ rational, $q > 0$ and $\gcd(p,q) = 1$
   - $F(x) = 0$ for x irrational
   - $F(0) = 1$

   Prove that

   (a) Given $\epsilon > 0$, the set $\{p/q, \quad 1/q \geq \epsilon, \quad p, \quad q \in I \quad \text{and} \quad p/q \in [0,1]\}$ is finite.

   (b) F is not continuous at any rational.

   (c) F is continuous at any irrational

   (d) F can be extended to a function g on $\mathbb{R}$ such that g is continuous at each irrational but not at any rational.

2. Let $f(x) = \dfrac{1}{1+x^2}$, $x \in [1, +\infty)$. Show that f is uniformly continuous on $[1, +\infty)$.

3. Suppose f is continuous real-valued function on $\mathbb{R}$ and that

$$\lim_{x \to +\infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$$

   Prove that f is uniformly continuous on $\mathbb{R}$.

4. Let $D = \{x_1, x_2, \cdots, x_n, \cdots\} \subset \mathbb{R}$ be an arbitrary countable set. define f by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if} \quad x = x_n, \quad n = 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

   Prove that f is continuous at each point outside D.

5. Let $f : \mathbb{R} \to \mathbb{R}$ have the property that

$$f(x+y) = f(x) + f(y) \quad \text{for all } x \text{ and } y \in \mathbb{R}$$

   Prove that if f is continuous at 0, then f is uniformly continuous.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function.Show that f can have at most countably points of discontinuity.