Problems: AIMS Electromagnetism

Many times in physics one wants to solve systems of ordinary second order differential equations (equations of motion for example). It is standard to try to put them into first order formalism by going to phase space. Once you get to this stage, you can try putting the system on a computer by evolving the equations of motion discretely. Many times this destroys certain aspects of the dynamics. However, if you do things right, you can get some things to work better then expected. We will illustrate this point with an example.

Lets consider two interacting particles that have the same mass (m) but opposite charge (q).

1. Derive Newton's second law of motion for these particles.

We will now describe things in *phase space* (for a description of phase space, go here http://en.wikipedia.org/wiki/Phase_space) To move to phase space, introduce the momenta

$$\vec{p}_i = m \frac{d}{dt} \vec{x}_i$$

where $i = 1, 2$ labels the two particles.

2. In terms of momenta and positions, show that Newton's second law becomes a set of first order differential equations.

3. A naive implementation of the evolution of the system is given by evolving

$$\vec{x}_i(t + \delta t) = \vec{x}_i(t) + \delta t \frac{\vec{p}_i(t)}{m}$$

and

$$\vec{p}_1(t + \delta t) = \vec{p}_1(t) + \delta t \frac{q^2}{4\pi\epsilon_0} \frac{\vec{x}_2(t) - \vec{x}_1(t)}{|\vec{x}_2(t) - \vec{x}_1(t)|^3}$$

$$\vec{p}_2(t + \delta t) = \vec{p}_2(t) + \delta t \frac{q^2}{4\pi\epsilon_0} \frac{\vec{x}_1(t) - \vec{x}_2(t)}{|\vec{x}_1(t) - \vec{x}_2(t)|^3}$$

These equations are not exact - explain how they were derived and when you expect them to be accurate. Show that the dynamics defined by these equations is not reversible: going backwards in time by changing $\delta t \to -\delta t$ does not get you exactly back to where you started. Implement these equations numerically and generate the solution given any initial positions and velocities. Choose a few different initial conditions and predict qualitatively what you expect the motion of the charged particles to be. Compare to the results obtained from your code. What can you say about $\vec{p}_1(t + \delta t) + \vec{p}_2(t + \delta t)$?

There is a better way to study this problem: you think of momenta as being evaluated at half times, and positions at full times. Thus we get

$$\vec{x}_i(t + \delta t) = \vec{x}_i(t) + \delta t \frac{\vec{p}_i(t + \frac{1}{2}\delta t)}{m}$$

and

$$\vec{p}_1(t + \tfrac{1}{2}\delta t) = \vec{p}_1(t - \tfrac{1}{2}\delta t) + \delta t \frac{q^2}{4\pi\epsilon_0} \frac{\vec{x}_2(t) - \vec{x}_1(t)}{|\vec{x}_2(t) - \vec{x}_1(t)|^3}$$

$$\vec{p}_2(t + \tfrac{1}{2}\delta t) = \vec{p}_2(t - \tfrac{1}{2}\delta t) + \delta t \frac{q^2}{4\pi\epsilon_0} \frac{\vec{x}_1(t) - \vec{x}_2(t)}{|\vec{x}_1(t) - \vec{x}_2(t)|^3}$$

4. Even though this looks almost identical to what we had before, show that the new equations are now time reversible (just send $\delta t \to -\delta t$ and do appropriate shifts to check that you really get back to where you started). Implement these equations numerically and generate the solution given any initial positions and velocities. What can you say about $\vec{p}_1(t + \frac{1}{2}\delta t) + \vec{p}_2(t + \frac{1}{2}\delta t)$?

This is called the leap-frog algorithm (for a description of this algorithm see http://en.wikipedia.org/wiki/Leapfrog_integration). For problems like the one above, it has rather nice properties. The most important one is that it preserves *Liouvilles theorem* (it keeps the volume element of phase space constant). (For a discussion of Liouville's theorem, go here http://en.wikipedia.org/wiki/Liouville's_theorem_(Hamiltonian) .)

5. For our problem this algorithm does something else that is quite amazing. For a two particle system like the one we are studying here, we can reduce it to a single particle problem by a clever choice of coordinates (see http://en.wikipedia.org/wiki/Reduced_mass). Rewrite the leap frog algorithm using the center of mass

$$\vec{X} = \frac{\vec{x}_1 + \vec{x}_2}{2}$$

and the relative coordinate

$$\vec{x} = \vec{x}_1 - \vec{x}_2$$

Show that the center of mass equations are trivial to solve, leaving only the problem for the relative coordinate. Implement the relative coordinate equation numerically and generate the solution given any initial position and velocity. Show (using your program is acceptable) that the relative coordinate motion is motion in a plane and the relative coordinate orbits the origin. Keplers second law - "sweeping equal areas in equal time intervals"

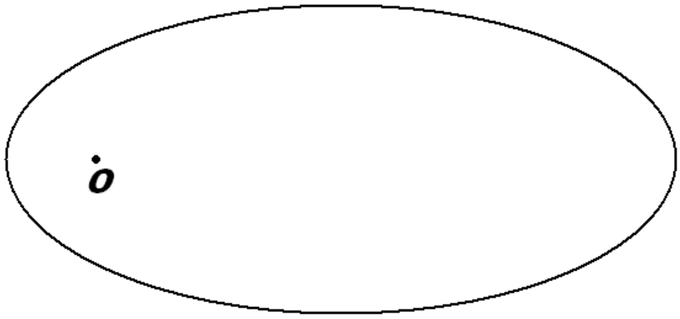

Fig. 1: The particle trajectory for question 6.

(see http://www-istp.gsfc.nasa.gov/stargaze/Skepl2nd.htm for a discussion) holds for our system. Find a proof that the above system sweeps equal areas in equal times around the origin $\vec{x}_1 - \vec{x}_2 = 0$.

6. Consider the trajectory given. The point O is the origin. Using Kepler's second law, describe how the velocity of the particle changes as it completes one orbit.

If you have a system with a lot of symmetries, sometimes the leap from algorithm will preserve a lot of these symmetries and also the *conserved quantities* so that you can evolve the system for much larger values of δt without loss of information.

What you should hand in:

Individual work: Write up and submit your solution to problem 5.

As a group: You need to submit a report, written in Latex, which answers the questions above. View your report as a document you might submit to a scientific journal for publication. Therefore, approach your report with as much professionalism as possible. Remember to cite your sources properly to avoid plagiarizing another scientist's work.

When writing a scientific report, remember that your purpose is to communicate your findings to the reader and to explain the research behind your findings. Organizing your thoughts in writing and clearly expressing your reults can be a valuable part of the learning experience.

Your report will be viewed with a critical eye - you are being critiqued for organization and your style of writing in this type of format. A good rule is to begin each paragraph/section with a sentence reflecting the topic of that paragraph/section.

You can use either a passive or an active voice in your report, although the active voice often reads as being more concise. The passive voice sometimes reads as a longer description of the same idea. For example: 'It was observed that the composition of the formula resulted in a more secure bond of the two layers in this experiment' (passive voice), compared with 'I/we observed a more secure bonding of the two layers in this experiment' (active voice).

Along the same lines, the more "to the point" your review is, the better; less is more, as far as word count goes. Overly long sentences are frustrating and confusing to the reader.

When writing about the results and your methods for achieving them, remember to use the past tense of verbs, since your findings have already been conducted and now you are writing about them. However, the report, the theory, and your equipment should be referred to in the present tense of the verb because they still exist.

Always proofread your work. Do not rely on your computer's spell-check feature to catch every spelling mistake or grammatical error.

Be careful not to be redundant (repeating the same words or thoughts or BOTH several times within a sentence or paragraph). This happens frequently in scientific papers because many report writers feel the need to stress their ideas, but in some cases, writers do so to the point of obsession. If you've said it once with accuracy, consider it said a thousand times.

Watch the use of articles (words like a, an, and the), as in many cases they are missing or added where they need not be. (An article refresher: An article basically "agrees" with a noun within a noun phrase; for example, "The black cat jumped from the tree" indicates a specific noun (the black cat); or, in the case of a or an, "An apple lay on the ground" indicates a reference to any member of this group (in this case apples in general). This is a general rule within the English language. The word the is therefore called a "definite" article, and words like a or an are called "indefinite" articles.)

Watch the use of words like and, so, and but. Many scientific writers often start a sentence with these words, which are called "conjunctions" in English grammar; don't make this a habit. (Conjunction refresher: A conjunction's purpose is to join two similar thoughts (words or phrases) within a sentence in order to "coordinate" them. Conjunctions are usually preceded by a comma to mark the "separation" of the thought or idea. It should now be clear that its not a great idea to keep starting sentences with conjunctions.)

When you should hand it in: The project is due on or before 17:30 on the 16 November.