AIMS - Nepomechie - Quantum Mechanics and Quantum Spin Chains

Problem Set #2 Due 6:00 p.m. December 12, 2009

For the things we have to learn before we can do them, we learn by doing them

- Aristotle (384 BC - 322 BC)

1. Show that

   (a) if $A = A(p)$ is a function of the operator p that can be expressed as a power series in p, then

   $$[x\,, A] = i\hbar \frac{\partial A}{\partial p}\,.$$

   (b)

   $$U(a)^\dagger\, x\, U(a) = x + a\,,$$

   where $U(a) = e^{-iap/\hbar}$ is the translation operator, and a is a real number (not an operator). *Hint:* use the Baker-Hausdorff lemma.

2. A particle is in a one-dimensional "infinite square well" potential

   $$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le L\,, \\ \infty & \text{otherwise} \end{cases}\,.$$

   Consider the eigenvalue problem

   $$H|\psi\rangle = E|\psi\rangle\,,$$

   which in coordinate space is

   $$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx'^2} + V(x')\right]\psi(x') = E\psi(x')\,.$$

   Determine all the possible normalized wavefunctions $\psi(x')$ (which must vanish both at $x' = 0$ and at $x' = L$) and the corresponding energies E.

3. Coherent states

   (a) Show that if $[A\,, B]$ is a c-number (i.e., it commutes with everything), then

   $$\left[A\,, e^B\right] = [A\,, B]\, e^B\,.$$

   Hint: use the Baker-Hausdorff lemma.

(b) Consider a one-dimensional harmonic oscillator. Define the state

$$|\psi_\beta\rangle = e^{\beta a^\dagger}|0\rangle\,,$$

where $|0\rangle$ is the ground state, and β is an arbitrary complex number. Use the above result (a) to show that $|\psi_\beta\rangle$ is an eigenket of the lowering operator a with eigenvalue β,

$$a|\psi_\beta\rangle = \beta|\psi_\beta\rangle\,.$$

Since a is not Hermitian, its eigenvalues do not have to be real.

(c) Let $|n\rangle$ be the usual normalized eigenket of the number operator $N = a^\dagger a$ with eigenvalue n. Compute the scalar products $\langle n|\psi_\beta\rangle$, and thus find the eigenket $|\beta\rangle$ of a, proportional to $|\psi_\beta\rangle$, that is normalized to unity,

$$a|\beta\rangle = \beta|\beta\rangle\,, \qquad \langle\beta|\beta\rangle = 1\,,$$

where $\langle\beta|$ is the bra which is dual to the ket $|\beta\rangle$.

(d) Since a is not Hermitian, its eigenkets with different eigenvalues do not have to be orthogonal. Let $|\beta\rangle$ and $|\gamma\rangle$ be two normalized eigenkets of a. Compute $|\langle\gamma|\beta\rangle|^2$.

(e) Compute the expectation value of the Hamiltonian in the state $|\beta\rangle$, i.e., $\langle\beta|H|\beta\rangle$.

4. Construct the ket $|\vec{S}\cdot\hat{n}\,;+\rangle$ such that

$$\vec{S}\cdot\hat{n}|\vec{S}\cdot\hat{n}\,;+\rangle = \frac{\hbar}{2}|\vec{S}\cdot\hat{n}\,;+\rangle\,,$$

where $\vec{S}$ is the spin operator for spin-1/2, and $\hat{n}$ is a unit vector with spherical coordinates α and β, i.e., with Cartesian coordinates given by

$$\hat{n} = (\cos\alpha\sin\beta\,, \sin\alpha\sin\beta\,, \cos\beta)\ .$$

Treat this as an eigenvalue problem. Verify that you get the expected results when $\hat{n}$ is $\hat{x}$, $\hat{y}$ and $\hat{z}$, i.e., for $(\alpha,\beta) = (0,\pi/2)\,,(\pi/2,\pi/2)\,,(*,0)$, respectively.

Hint: Use the identity $1-\cos\beta = 2\sin^2\frac{\beta}{2}$.

5. Spin matrices

(a) Construct a set of three 4×4 matrices corresponding to the spin operators $\vec{S}$ for spin $s = 3/2$. Use *Sage* to explicitly verify that

$$[S_i\,, S_j] = i\hbar\sum_k \epsilon_{ijk}S_k\,.$$

You may set $\hbar = 1$.

(b) Repeat for spin $s = 2$, for which case the matrices are instead 5×5.

6. Tensor products

   (a) Use *Sage* to compute, for the case of two spin-1/2 particles,

   $$\mathcal{S}_i = S_i \otimes I + I \otimes S_i \,,$$

   where S_i are the usual 2×2 spin-1/2 spin matrices, and I is the 2×2 identity matrix. Make sure you obtain the result stated in lecture!

   (b) Again using *Sage*, verify that the matrices you obtained in part (a) satisfy

   $$[\mathcal{S}_i \,, \mathcal{S}_j] = i\hbar \sum_k \epsilon_{ijk} \mathcal{S}_k \,.$$

   You may set $\hbar = 1$.

   (c) Again using *Sage*, construct $\vec{\mathcal{S}}^2$. Find a set of 4 orthonormal kets which are simultaneous eigenkets of both $\vec{\mathcal{S}}^2$ and $\mathcal{S}_z$. Summarize your results in a table containing each of the kets and the corresponding values of s and m.

   (d) Verify (again using *Sage*) that the matrix U given by

   $$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

   is unitary, and that the matrices

   $$U^\dagger \, \mathcal{S}_i \, U$$

   are block-diagonal corresponding to spin values $s = 1$ and $s = 0$.

7. Add angular momenta $s_1 = 1$ and $s_2 = 1$ to form states with spin $s = 2, 1, 0$. Construct all (nine) $|s \,, m\rangle$ eigenkets, writing your answer as

   $$|s = 1 \,, m = 1\rangle = \frac{1}{\sqrt{2}} \left(|1 \,, 0\rangle - |0 \,, 1\rangle \right) \,, \ldots \,,$$

   where $|1 \,, 0\rangle$ is a short-hand notation for $|s_1 = 1 \,, m_1 = 1\rangle \otimes |s_2 = 1 \,, m_2 = 0\rangle$, etc.