AIMS - Nepomechie - Quantum Mechanics and Quantum Spin Chains

Problem Set #1 Due 6:00 p.m. December 5, 2009

All things are difficult before they are easy

- Thomas Fuller (1608 - 1661)

1. Show that

(a)
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\omega\lambda}\frac{n}{\lambda^2+\frac{n^2}{4}} = e^{-n|\omega|/2}\,, \qquad n>0\,,$$

(b)
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\lambda}\frac{1}{\cosh\frac{\omega}{2}} = \frac{1}{\cosh(\pi\lambda)}\,,$$

(c)
$$\int_{-\infty}^{\infty} d\omega\, \frac{e^{-|\omega|/2}}{2\cosh\frac{\omega}{2}} = 2\ln 2\,.$$

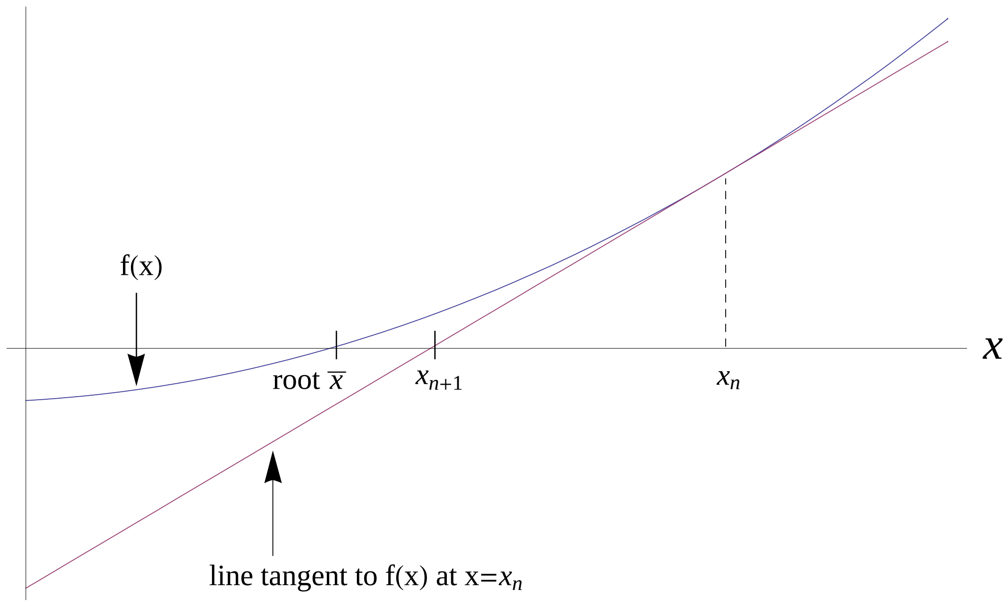
Hint: For (a) and (b), use contour integration.

2. Given a function $f(x)$, a nice and simple iterative way of finding a root $\bar{x}$ (i.e., $f(\bar{x})=0$) starting from some initial guess x_0, is "Newton's method":

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\,.$$

It can readily be understood from the following figure, noting that the line which is tangent to $f(x)$ at $x=x_n$ has slope

$$f'(x_n) = \frac{\text{“rise”}}{\text{“run”}} = \frac{f(x_n)}{x_n - x_{n+1}}\,.$$



It can also be understood from Taylor's series

$$0 \approx f(x_{n+1}) \approx f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

Write a program in *Sage* which implements Newton's method. Loop **while** the "error" is greater than 10^{-13} (think about how to define "error"!) and the number of iterations is less than 10. Use Jan's test function

$$f(x) = x^2 - 3\cos(x) + e^x ,$$

taking as your initial guess $x_0 = 1.0$. What is your value for the root $\bar{x}$? How many iterations did it take?

3. Recall that the angular momentum $\vec{L}$ of a point particle is given by $\vec{L} = \vec{x} \times \vec{p}$, where $\vec{x}$ is the position and $\vec{p}$ is the momentum. That is,

$$L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} x_j p_k , \qquad i = 1, 2, 3 .$$

Prove the following Poisson bracket identities:

(a) $\{L_i \, , x_j\} = \sum_{k=1}^{3} \epsilon_{ijk} \; x_k$

(b) $\{L_i \, , p_j\} = \sum_{k=1}^{3} \epsilon_{ijk} \; p_k$

(c) $\{L_i \, , L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} \; L_k$

Hint for part (c): You may assume the identity

$$\sum_{k=1}^{3} \epsilon_{ijk} \, \epsilon_{mnk} = \delta_{im} \, \delta_{jn} - \delta_{in} \, \delta_{jm} .$$

4. Consider a point particle in three dimensions in a central potential, with Lagrangian:

$$L = \frac{1}{2}m\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt} - V(r)\,, \qquad r = \sqrt{\vec{x}\cdot\vec{x}}\,.$$

   (a) Write Lagrange's equations for this system.

   (b) Show explicitly (using the equations of motion) that the angular momentum $\vec{L}$ for this system is conserved, i.e.,

$$\frac{d}{dt}L_i = 0\,, \qquad i = 1, 2, 3\,.$$

   (c) What is the Hamiltonian H for this system?

   (d) Show explicitly (with the help of the first two identities in Problem 3 above) that the Poisson bracket of the Hamiltonian with each of the components of $\vec{L}$ vanishes,

$$\{H\,, L_i\} = 0\,, \qquad i = 1, 2, 3\,.$$

5. Show that, for linear operators X, Y, Z,

   (a) $[X\,, YZ] = [X\,, Y]\,Z + Y\,[X\,, Z]$

   (b) $(XY)^\dagger = Y^\dagger X^\dagger$

6. Let B be the Hermitian matrix

$$B = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}\,.$$

   (a) Determine (by hand) the eigenvalues and corresponding orthonormal eigenvectors of this matrix.

   (b) Try to check your answer using *Sage*.

   (c) Use your result from part (a) to construct the matrix U which diagonalizes B. Verify with *Sage* that U is unitary, and that

$$U^\dagger\,B\,U$$

   is a diagonal matrix given by the eigenvalues of B.

7. Consider a 3-dimensional vector space. If the set of orthonormal kets $\{|1\rangle\,, |2\rangle\,, |3\rangle\}$ is used as a basis, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\,, \qquad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\,,$$

with a and b both real and nonzero.

(a) Obviously A has a degenerate spectrum. Does B also have a degenerate spectrum?

(b) Show that A and B commute.

(c) Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of the pair of eigenvalues completely characterize each eigenket?

Hint for part (c): It should be clear that the vector $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ is one of the normalized kets which is a simultaneous eigenket of both A and B, with eigenvalues a and b, respectively. Hence, we can specify this vector by the ket $|a,b\rangle$. That is,

$$|a,b\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \tag{1}$$

Now your job is to find the two remaining kets!

8. For any operators A and B, prove the famous Baker-Campbell-Hausdorff formula

$$e^{A}\,B\,e^{-A} = B + [A\,,B] + \frac{1}{2}\,[A\,,[A\,,B]] + \ldots .$$

Hint: Consider the operator

$$B(t) \equiv e^{At}\,B\,e^{-At} .$$

By differentiating with respect to t, obtain a first-order differential equation for $B(t)$. Solve this equation by iteration, and then set $t = 1$.