1. The Gravitational Constant

As far as we can tell, all ancient societies developed something we would recognize as astronomy. Even predicting how celestial bodies move in the future, based on accurately modelling what they have done before and then extrapolating it, has been around for at least two thousand years.[1] Kepler's laws from circa 1630 were part of this tradition.

What Newton did in 1666 or thereabouts, however, was something very different. He derived Kepler's laws from physical principles. It was the beginning of astrophysics.

So let us begin this course by going back to what Newton did in the 17th century, and looking at it with our 21st century eyes.

§1.1 TWO GRAVITATING BODIES. The central physical postulate in Newtonian gravity is that any two particles experience an attractive force towards each other, proportional to their masses and inversely proportional to the square of their mutual distance.

To express the above in equations, we consider two bodies: one with mass m_1, position $\mathbf{r}_1$ and velocity $\mathbf{v}_1$, and the second with m_2, $\mathbf{r}_2$, $\mathbf{v}_2$. Let $\mathbf{r}$ be the difference $\mathbf{r}_2 - \mathbf{r}_1$, and similarly $\mathbf{v}$. The equations of motion are

$$m_1\dot{\mathbf{v}}_1 = -m_2\dot{\mathbf{v}}_2 = \frac{Gm_1m_2}{r^2}\,\hat{\mathbf{r}} \tag{1.1}$$

Since the forces are equal and opposite, clearly $m_1\dot{\mathbf{v}}_1 + m_2\dot{\mathbf{v}}_2 = 0$, meaning that the barycentre has no acceleration. The rest of the content of (1.1) is expressed by an effective one-body problem

$$\dot{\mathbf{v}} = -\frac{\mu}{r^2}\,\hat{\mathbf{r}} \qquad \mu \equiv G(m_1 + m_2) \tag{1.2}$$

If one of the bodies is negligibly small, it orbits around the more massive body. But in general, both bodies orbit around the common barycentre.

Note that (1.1) is postulated only for point masses. However, the gravitational interaction of a spherical mass with external bodies is as if the mass were at the central point. This can be proved either by integrating the gravitational force over the surface of a sphere, or more concisely, by adapting the derivation of Gauss's flux law in electrostatics. But here we skip the proof and assume that (1.1) and (1.2) hold for spherical masses.

PROBLEM 1.1. Write down a circular-orbit solution to (1.2). Hence show that the orbital time just above a planet of mean density ρ is

$$\sqrt{\frac{3\pi}{G\rho}}$$

[1] For a beautiful example, look up *Antikythera mechanism*.

§1.2 UNITS OF THE GRAVITATIONAL CONSTANT. In SI units

$$G = 6.67300 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}} \tag{1.3}$$

This expression doesn't tell us very much, except maybe that gravity is weak. A more interesting way to express G is

$$G^{-\frac{1}{2}} = 1.075\ \left(\mathrm{g\,cm^{-3}}\right)^{\frac{1}{2}}\ \mathrm{hr} \tag{1.4}$$

This immediately tells us that orbital time scales depend on density. Moreover, for a solid body (ice, rock) the orbital speed is of the order of the speed of the minute hand in a clock of that size. Imagine the centrifugal force on the minute hand of a clock, the gravitational force of the clock body will be of the same order. Hence very small, but something we can imagine measuring ourselves.

Another way of writing G is

$$GM_\odot \simeq 4\pi^2\,\mathrm{au^3\,yr^{-2}} \tag{1.5}$$

As we will verify below, this is just a statement that the Earth follows a Newtonian two-body orbit. If we substitute $M_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$, $\mathrm{au} = 1.496 \times 10^{11}\,\mathrm{m}$, $\mathrm{yr} = 3.156 \times 10^{7}\,\mathrm{s}$, we recover (1.3) to three significant digits, but not perfectly, because the Earth's orbit is perturbed. The form (1.5) also hints at another important fact: the product $GM_\odot$ is known much more accurately that G or $M_\odot$ separately. For gravitational dynamics, it is often useful to define the Sun as the mass-standard; in fact, for spacecraft dynamics it is essential.

PROBLEM 1.2. The sun is about half a degree on the sky. Estimate the density of the sun in $\mathrm{g\,cm^{-3}}$.

§1.3 KEPLER'S LAWS. There are many ways of deriving Kepler's laws from the equations of motion (1.2). Here is one.[2]

The third law is the easiest, and in fact it is just one of various possible scaling relations. To derive it, let us multiply all lengths by (a constant) λ_L, all masses by λ_M, and all times by λ_T. This is just a change of units, no physics has been changed. Incorporating these factors into the equation of motion (1.2) gives

$$\dot{\mathbf{v}} = -\lambda_M \lambda_L^{-3} \lambda_T^2 \frac{\mu}{r^2} \hat{\mathbf{r}} \tag{1.6}$$

Clearly, the equation of motion is invariant if

$$\lambda_L^2 = \lambda_T^3 \qquad \lambda_M = 1 \tag{1.7}$$

which is Kepler's third law.

[2] To get a flavour of how Newton did it, see the book *Feynman's Lost Lecture*.

To derive the second law, we consider

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{v} \tag{1.8}$$

that is, the angular momentum per unit mass, which has the geometrical interpretation of rate of area swept by $\mathbf{r}$.

If we take the time derivative of $\mathbf{L}$, there are two terms: one is $\mathbf{v} \times \mathbf{v}$ and is clearly zero; the other is $\mathbf{r} \times \dot{\mathbf{v}}$, and since $\dot{\mathbf{v}}$ is in the direction of $\mathbf{r}$, this term is also zero. Hence $\mathbf{L}$ is conserved, which is Kepler's second law.

Now we consider the so-called Runge-Lenz vector

$$\mathbf{e} \equiv \mathbf{v} \times \mathbf{L}/\mu - \mathbf{r}/r \tag{1.9}$$

We now (1.2) and sorting through the vector algebra, noting in particular that $\dot{r} \neq v$ but rather $\dot{r} = \mathbf{r} \cdot \mathbf{v}/r$, we derive that $\mathbf{e}$ is also conserved. Taking $\mathbf{e} \cdot \mathbf{r}$ and rearranging we have

$$r = \frac{L^2/\mu}{1 + e \cos f} \tag{1.10}$$

where f denotes the angle between $\mathbf{e}$ and $\mathbf{r}$. Equation (1.10) is the equation for a conic section with e being the eccentricity. We thus have Kepler's first law.

It is evident from (1.9) that $\mathbf{e}$ must be in the orbital plane, and from (1.10) it follows that $\mathbf{e}$ points towards the pericentre. To get the dynamical meaning of e, let us introduce

$$E = \frac{v^2}{2} - \frac{\mu}{r} \tag{1.11}$$

the total energy. Now taking the square of equation (1.9)—recall for this that $\mathbf{v}$ is perpendicular to $\mathbf{L}$—we derive

$$1 - e^2 = -\frac{2EL^2}{\mu^2} \tag{1.12}$$

We can now rewrite the orbit shape (1.10) slightly, eliminating L. Introducing the semi-major axis a, defined by

$$E = -\frac{\mu}{2a} \tag{1.13}$$

we have

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \tag{1.14}$$

The above derivation is valid for both bound ($E < 0$) and unbound ($E > 0$) orbits, but in the latter case a is negative and does not have a simple geometric interpretation.

PROBLEM 1.3. Imagine a scale model of the solar system, with the Earth scaled down to the size of a small stone and all lengths similarly, but with all densities remaining the same. This scale model is then put in space somewhere far from external gravitational fields.

The orbital times in this model solar system will be the same as in the real solar system. Why?

§1.4 The orbits in time. Computing the orbit as a function of time is a little more subtle. To derive this, we first introduce a new variable τ defined by

$$dt = r\,d\tau \tag{1.15}$$

Let primes denote derivatives with respect to τ. It follows that

$$r' = \mathbf{r}\cdot\mathbf{v} \qquad \mathbf{r}' = r\mathbf{v} \qquad \mathbf{v}' = -\mu\mathbf{r}/r^2 \tag{1.16}$$

Using these little identities we derive

$$r'' - 2Er = \mu \tag{1.17}$$

We can choose the boundary condition $r = a(1-e)$ at $\tau = 0$ without loss of generality. Introducing the variable

$$\eta = \sqrt{-2E}\,\tau \tag{1.18}$$

we have

$$r = a(1 - e\cos\eta) \tag{1.19}$$

Substituting into 1.15) and solving gives

$$t = \left(a^3/\mu\right)^{\frac{1}{2}}(\eta - e\sin\eta) \tag{1.20}$$

Thus, $r(t)$ is expressed implicitly in terms of η. There is no explicit expression in terms of elementary functions. Meanwhile, the angular position is still given by (1.14).

The expression for (1.19) is clearly periodic, whenever η changes by 2π. Hence from (1.20) we have

$$t_{\rm orb} = 2\pi\left(a^3/\mu\right)^{\frac{1}{2}} \tag{1.21}$$

and is independent of eccentricity.

A sphere collapsing under gravity follows the same equations as two mass points coming together. For this case, we put $e = 1$ in the time solution (1.20). The free-fall time $t_{\rm ff} = \frac{1}{2}t_{\rm orb}$. Putting $R = 2a$ and $R^3 = 3/(4\pi)V$ we have

$$t_{\rm ff} = \left(\frac{3\pi}{32G\rho}\right)^{\frac{1}{2}} \tag{1.22}$$

§1.5 Gravitational focusing. Here is an interesting example involving unbound orbits.

Writing L as bv_∞, where b is the impact parameter and v_∞ is the speed at infinity, (1.12) becomes

$$e^2 = 1 + 4\frac{E^2b^2}{\mu^2}$$

Writing $p = a(1-e)$ for the pericenter, (1.14) becomes

$$e = 1 + 2\frac{Ep}{\mu}$$

Comparing these two equations we have

$$b^2 = p^2\left(1 + \frac{\mu}{pE}\right) \tag{1.23}$$

This expression is important in planet formation, and the term in brackets is known as the Safronov number. The interpretation is as follows. Suppose a protoplanet has radius p. A planetesimal with impact parameter p will obviously collide with it. But (1.23) shows that larger impact parameter will also lead to collision. The enhancement term $\mu/(pE)$ is the ratio to potential energy at the surface to the total energy. Alternatively, we can think of it $v^2_{\rm esc}/v^2_\infty$.

PROBLEM 1.4. Look up this animation of stars near the Galactic center:

http://www.astro.ucla.edu/~ghezgroup/gc/images/2008orbits_animfull.gif

These observations (from 1995 to the present, extrapolated to 2010) show the stars to be on precisely Keplerian orbits around a supermassive central object, presumably a black hole.

Make a rough estimate of the mass of the black hole, based on this animation. The only additional information needed is that the Galactic centre is about 8 kpc away, meaning that the bar marked $0.1''$ corresponds to a distance of 800 au.

2. The three-body problem

Having solved the two-body problem, Newton wanted to calculate the perturbations due to other bodies. He was especially interested in calculating the tidal effect of the Sun on the Moon's orbit. This proved very difficult, and we find Newton complaining in a letter to Halley that the Moon's orbit made his head ache and kept him awake at night…

There are two reasons for Newton's headache. One is that the Moon is comparatively weakly bound to the Earth, and consequently perturbations have a rather large effect. The second reason is that there is no general solution to the gravitational three-body problem (though various particular cases are soluble). Progress could, however, be made with perturbation theory and numerics, and from Newton's time till around 1920, the three-body problem pretty much was astrophysics.

In this chapter we will consider the so-called restricted three-body problem. This has two massive bodies in circular Keplerian orbits, while a third body of negligible mass moves under their combined gravity in the common plane. The restricted problem captures most of the interesting qualitative features of the general three-body problem. In particular—as Poincaré showed around 1900 but which became well known only in the 1960s, when computers became common—it contains chaos. We will not attempt perturbation theory. Instead, we will try to gain insight using Hamiltonian dynamics and numerical orbit integration.

§2.1 HAMILTONIAN FOR THE RESTRICTED THREE-BODY PROBLEM. To simplify the equations a little, we choose the unit of length to be the distance between the two massive bodies, and the unit of time to be such that G times the total mass is unity. The two main bodies will then rotate with unit angular velocity.

Let us write ϵ for the mass of one massive body relative to the total (ϵ does not have to be small, just between 0 and 1). We take coordinates rotating about the barycenter, such that the massive bodies are at $(-\epsilon, 0)$ and $(1-\epsilon, 0)$.

The equations can be expressed by the Hamiltonian

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + V(x,y) + yp_x - xp_y \tag{2.1}$$

where

$$V = -\frac{1-\epsilon}{\sqrt{(x+\epsilon)^2 + y^2}} - \frac{\epsilon}{\sqrt{(x-1+\epsilon)^2 + y^2}} \tag{2.2}$$

In (2.1) the $yp_x - xp_y$ part (which is minus angular momentum) reflects a coordinate system rotating with unit angular velocity. To see why, consider Hamilton's equations for (2.1)

$$\begin{aligned} \dot{x} &= p_x + y \qquad & \dot{p}_x &= -\frac{\partial V}{\partial x} + p_y \\ \dot{y} &= p_x - x \qquad & \dot{p}_y &= -\frac{\partial V}{\partial y} - p_x \end{aligned} \tag{2.3}$$

Differentiating the first column of equations and eliminating p_x, p_y gives

$$\begin{aligned}\ddot{x} &= -\frac{\partial V}{\partial x} + 2\dot{y} + x \\ \ddot{y} &= -\frac{\partial V}{\partial y} - 2\dot{x} + y\end{aligned} \tag{2.4}$$

In each of the equations here, the rightmost term is the centrifugal force, and the term before that is the Coriolis force.

PROBLEM 2.1. In the restricted three-body Hamiltonian (2.1) let us put

$$x = 1 - \epsilon + X$$

and assume ϵ, X, y are all small. Show that in this regime, the equations of motion (2.4) simplify to

$$\begin{aligned}\ddot{X} &= -\epsilon\frac{X}{R^3} + 2\dot{y} + 3X \\ \ddot{y} &= -\epsilon\frac{y}{R^3} - 2\dot{X}\end{aligned}$$

where $R \equiv \sqrt{X^2 + y^2}$. These are known as Hill's equations.

Hint: The expansion

$$(1+\alpha)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}\alpha + \tfrac{3}{8}\alpha^2 + O(\alpha^3)$$

is useful here.

§2.2 ROCHE LOBES. Let us introduce the function

$$V_R(x,y) = V(x,y) + \tfrac{1}{2}(x^2 + y^2) \tag{2.5}$$

which lets us rewrite (2.4) as

$$\begin{aligned}\ddot{x} &= -\frac{\partial V_R}{\partial x} + 2\dot{y} \\ \ddot{y} &= -\frac{\partial V_R}{\partial y} - 2\dot{x}\end{aligned} \tag{2.6}$$

In other words, V_R is an effective potential that incorporates the centrifugal force but not the Coriolis force.

Figure 2.1 shows contours of V_R, and we can infer some qualitative features of the orbit of the third body simply be examining this figure. The two shaded regions (known as Roche lobes) are regions with orbits going around one of the two main bodies. The points L1 to L5 (known as the Lagrange points) are equilibrium points, where V_R exerts no force. The equilibria, however, are not all stable. L1, L2, and L3 are saddle points, where the third body is unstable to moving away in the x direction. L4 and L5 are maxima of V_R, which suggests that they are also unstable. But L4 and L5 are stabilized

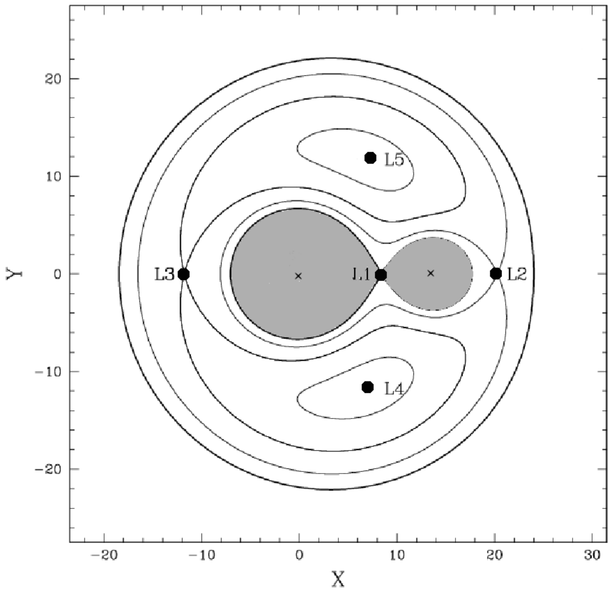

Figure 2.1: Contours of the function $V_R(x,y)$. The two main bodies are marked by $\times$ and there V_R goes to $-\infty$. L1, L2 and L3 are saddle points, while L4 and L5 are maxima. The Roche lobes are shaded; these lobes are where V_R is lower than at L2.

by the Coriolis force. To see why, consider (2.6) without the V_R terms. Writing v_x for $\dot{x}$ and v_y for $\dot{y}$ we have

$$\begin{aligned} \dot{v}_x &= 2v_y \\ \dot{v}_y &= -2v_x \end{aligned} \tag{2.7}$$

In other words, the Coriolis force on its own acts like a harmonic oscillator for the velocities.

§2.3 Surfaces of section. A surface of section (also called a Poincaré section) is a slice through phase space illustrating features of the dynamics. It is simplest to define operationally.

Let us choose an energy E and start an orbit at $(x, y=0, p_x, p_y)$ with p_y given implicitly by

$$H(x, 0, p_x, p_y) = E \tag{2.8}$$

Since the Hamiltonian (2.1), is quadratic in p_y, (2.8) will have two solutions for p_y. For some regions of the plane (x, p_y) the solutions for p_y will be real, for the rest complex. These regions will be bounded by the curves

$$H(x, 0, p_x, 0) = 0 \tag{2.9}$$

Suppose now that p_y does have two real solutions. We choose the larger one (which implies $\dot{y} > 0$ and then follow the orbit starting from these initial conditions. Integrating the orbit for a long time, we generate the set of points

$$\mathcal{C} = \left\{(x, p_x) : \; y = 0, \, \dot{y} > 0\right\} \tag{2.10}$$

A plot of $\mathcal{C}$ gives more qualitative information about the orbit. For a start, if the orbit is periodic $\mathcal{C}$ will be a single point or a few discrete point, if the orbit is aperiodic but non-chaotic orbit $\mathcal{C}$ will be a curve, if the orbit is chaotic $\mathcal{C}$ will fill a region of the plane (x, p_x).

PROBLEM 2.2. [Open-ended] Numerically integrate Hamilton's equations for (2.1), and use plots of the orbits and their surfaces of section to explore different types of orbits. Suggested parameters: $\epsilon = 0.1, E = -2$.

3. Schwarzschild's spacetime

By 1860, observations of planetary orbits and calculations of planetary perturbations were already so advanced that a discrepancy between Mercury's orbit and Newtonian gravity at the level of $\sim 10^{-7}$ was measured. Theory seemed to predict an extra planet inner to Mercury. This fictitious planet Vulcan gained several claimed discoveries over the next few decades, and survives to this day in science folklore.

But meanwhile, developments in other areas of physics implied that a new theory of gravity was needed (of which Newtonian gravity would be a limit). Einstein provided one—general relativity—in 1916, and seemingly miraculously, it explained the discrepancy with Mercury's orbit, the first of its many successes.

The equations for the gravitational field in general relativity are too complicated for this course. But the orbit equations in a given gravitational field are not very much more difficult than in Newtonian gravity. We will study one interesting and important case, the Schwarzschild spacetime, which is the gravitational field around a spherical mass—which could be the Sun or the Earth, or a black hole.

§3.1 The Hamiltonian in a Schwarzschild spacetime. In spacetime, time is no longer an independent variable, but one of the coordinates. The role of the independent variable is taken by an abstract variable called the affine parameter.[3] In other words, we have

$$H(t, x, y, z, p_t, p_x, p_y, p_z)$$

In the usual notation of general relativity the Hamiltonian is $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$, but we need not worry about that, because for the Schwarzschild spacetime, H takes the more understandable form

$$H = -\frac{r/2}{r-2} p_t^2 + \tfrac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{(xp_x + yp_y + zp_z)^2}{r^3} \tag{3.1}$$

The unit of length has been chosen to be GM/c^2 and the unit of time such that $c = 1$. This Hamiltonian covers both ordinary matter and light: matter has $H < 0$ and light has $H = 0$.[4]

We can simplify further by considering Hamilton's equations. First, we have $\dot{p}_t = 0$. The constant value of p_t is not physically important, it simply sets the units of the affine parameter. A convenient value is $p_t = -1$. The equation for time is

$$\dot{t} = \frac{r}{r-2} \tag{3.2}$$

[3] Important: in this chapter, dot denotes derivative with respect to the affine parameter, not time.

[4] What $H > 0$ applies to, we can only speculate.

Second, if z and p_z are both zero initially they will stay zero. This is just a statement that the orbit is in a plane. So we can discard z and p_z without loss of generality. With these simplifications we have

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 - \frac{r/2}{r-2} - \frac{(xp_x + yp_y)^2}{r^3} \tag{3.3}$$

In polar variables we have

$$H = \frac{p_r^2}{2} + \frac{p_\phi^2}{2r^2} - \frac{r/2}{r-2} - \frac{p_r^2}{r} \tag{3.4}$$

If we are interested in understanding the shapes of orbits but not the details of orbital time, we can make a further simplification by disregarding the time equation (3.2) altogether, since t does not enter into the other Hamilton equations.

§3.2 The Newtonian limit. At large r, the third term in (3.3) or (3.4) is

$$-\tfrac{1}{2} - \frac{1}{r} - \frac{2}{r^2} - \frac{4}{r^3} - \dots$$

while the t equation (3.2) becomes

$$\dot{t} = 1$$

In this limit we thus have Newtonian orbits with t equal to the affine parameter, provided speeds are much slower than light. (What happens at large r for light we will see below.)

As r becomes smaller, we will see increasingly larger deviations from Newtonian, until the equations become singular at $r = 2$. This last is known as the Schwarzschild horizon.

§3.3 Circular orbits. Consider the polar form (3.4) of the Hamiltonian. Let us write p_ϕ^2 (which is constant) as $a^3/(a-2)^2$ where a is some constant. Then Hamilton's equation become

$$\begin{aligned} \dot{p}_r &= \frac{a^3}{(a-2)^2 r^3} - \frac{1}{(r-2)^2} - 2\frac{p_r^2}{r^3} \\ \dot{r} &= p_r\left(1 - \frac{2}{r}\right) \end{aligned} \tag{3.5}$$

We see that $r = a$ is a fixed point of these equations, corresponding to a circular orbit. Now consider

$$\left.\frac{d\dot{p}_r}{dr}\right|_{r=a} = \frac{(6-a)}{a(a-2)^3} + \frac{6p_r^2}{a^4} \tag{3.6}$$

Below $a = 6$ the circular orbits become unstable.

PROBLEM 3.1. Calculate the value of H for a circular orbit of radius a. Check that the large-a limit is $-\frac{1}{2} - 1/(2a)$. Something interesting happens at $a = 3$. Interpret that something physically.

§3.4 DEFLECTION OF PASSING TRAJECTORIES. Now we take the "cartesian" form (3.4) of the Hamiltonian, and consider the equation

$$\dot{p}_x = -\frac{x}{r(r-2)^2} + \frac{2p_x(xp_x + yp_y)}{r^3} - \frac{3x(xp_x + yp_y)^2}{r^5}$$

Now we put

$$x \simeq b \gg 1 \qquad p_x \simeq 0 \qquad p_y \simeq \begin{cases} 0 & \text{slow bodies} \\ 1 & \text{light} \end{cases}$$

which gives us

$$\dot{p}_x \simeq -\frac{b}{(b^2 + y^2)^{3/2}} - \frac{3by^2}{(b^2 + y^2)^{5/2}} \tag{3.7}$$

with the last term appearing for light only. Integrated over y, each term contributes $-2/b$.

Thus, if the impact parameter b is large, matter gets deflected by $2/b$ and light by $4/b$. This was the first prediction of general relativity to be tested, by a famous observation by Eddington and others in 1919. This appears to have been the key event that first made Einstein a household name.

PROBLEM 3.2. Numerically integrate Hamilton's equations for the "cartesian" form (3.3) of the Hamiltonian, plotting orbits and surfaces of section. (This is actually easier than for the restricted three-body problem.) Suggested value: $E = -0.51$.

4. Quantum processes

The development of quantum mechanics from 1900 onwards changed astrophysics in a way that could not been imagined at the end of the 19th century. Very quickly, it became possible to study the microphysics of stars. And from being pure gravity, astrophysics changed to being about the interaction of gravity and microphysics, leading to an understanding of stellar structure and physical cosmology.

We will not try to summarize quantum processes here, just remind ourselves of a few essential results before moving on.

§4.1 PLANCKIAN UNITS. In relativity one often works in units where $c = 1$, and we have tacitly already done this. In particle physics it is common to use units with $\hbar = c = 1$. Planckian units are more drastic still: we put $G = \hbar = c = 1$.

Planckian units are no more than a special choice of units for mass, length, and time: marbles, laps, and ticks[5] instead of kilograms, metres and seconds.

$$\begin{aligned} \text{marb} &\equiv (\hbar c/G)^{\frac{1}{2}} = 2.176 \times 10^{-8}\,\text{kg} \\ \text{lap} &\equiv \left(\hbar G/c^3\right)^{\frac{1}{2}} = 1.616 \times 10^{-35}\,\text{m} \\ \text{tick} &\equiv \left(\hbar G/c^5\right)^{\frac{1}{2}} = 5.392 \times 10^{-44}\,\text{s} \end{aligned} \tag{4.1}$$

Then the speed of light is 1 lap tick^{-1} and so on. From now on, we will work mainly in Planckian units. But we can convert to SI whenever we want, by multiplying by appropriate powers of marb, lap and tick.

The main physical constants that will appear are the nucleon mass m_N, the electron mass m_e, and the fine-structure constant α (which is $e^2/(4\pi\epsilon_0\hbar c)$ in SI). In Planckian units these are as follows.

$$\begin{aligned} m_N &= 7.688 \times 10^{-20} \\ m_e &= 4.186 \times 10^{-23} \\ \alpha &= 1/137.04 \end{aligned} \tag{4.2}$$

Planckian units makes formulas more concise, but it can make them look strange. For example, in the next chapter, we will encounter a mass which equals m_N^{-2}. If we multiply m_N^{-2} by marb^3, we get an expression with the dimensions of mass, namely $(\hbar c/G)^{\frac{3}{2}} m_N^{-2}$. To evaluate the mass in SI units, however, we can take a shortcut and simply multiply by the value of marb, which gives 3.68×10^{30} kg. In fact (and not coincidentally) it is the mass scale of stars.

PROBLEM 4.1. The Compton wavelength of the electron is defined as $2\pi/m_e$. What is its value in metres? What would be the expression if not using Planck units?

[5] Planckian units of mass, length, and time do not have standard names, so I have just given them some names.

Sometimes, we will want a temperature in Kelvin. But since temperature is basically a measure of energy per particle, we can interpret Kelvin as an energy scale. The value is

$$\mathrm{K} = 7.064 \times 10^{-33} \tag{4.3}$$

PROBLEM 4.2. Evaluate K in SI units of energy. You can check your answer by looking up "Boltzmann's constant".

PROBLEM 4.3. A common (but non-SI) unit of energy, often used to express the energy of individual particles, is the electron volt: $\mathrm{eV} = 8.196 \times 10^{-29}$ in Planckian units.

Express 1 eV (i) as an energy in SI units, (ii) as a temperature in K.

§4.2 QUANTUM IDEAL GASES. We need to take over some formulas from the theory of statistical thermodynamics. In fact, we really need only one formula, which looks like this

$$n(p)\,dp = \frac{g}{2\pi^2}\frac{p^2\,dp}{e^{(E-\mu)/T} \pm 1} \tag{4.4}$$

and gives the number density of gas particles as a function of momentum p. The plus sign refers to the so-called Fermi-Dirac particles (which includes electrons and protons), while the minus sign is for Bose-Einstein particles (mostly importantly photons). Here T is the temperature, assumed constant. The constant g is a small integer (typically $g = 2$) that depends on the detailed physics of the type of particle being considered, and is known as the degeneracy. The constant μ is known as the chemical potential, even if no chemistry is involved; it also depends on microphysics.

The total number density and energy density are

$$n = \int n(p)\,dp \qquad U = \int E(p)\,n(p)\,dp \tag{4.5}$$

§4.3 THE MAXWELL-BOLTZMANN GAS. The $E \gg \mu$ regime of the Bose-Einstein ideal gas (4.4) corresponds to the classical idea gas. Integrating

$$n(p)\,dp = \frac{g}{2\pi^2}\,e^{(\mu-E)/T}\,p^2\,dp \qquad E = \frac{p^2}{2m} \tag{4.6}$$

we derive

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{\mu/T} \tag{4.7}$$

§4.4 PHOTON GAS. The case $g = 2$, $\mu = 0$ and $E = p$ in a Bose-Einstein ideal gas corresponds to a photon gas, also known (for historical reasons) as blackbody radiation.

$$n(p)\,dp = \frac{1}{\pi^2}\frac{p^2\,dp}{e^{p/T} - 1} \tag{4.8}$$

The integrals are

$$n = \frac{2\zeta(3)}{\pi^2} T^3 \simeq 0.2435 T^3 \tag{4.9}$$

and

$$U = \frac{\pi^2}{15} T^4 \tag{4.10}$$

The pressure and irradiance are

$$P = \tfrac{1}{3} U \qquad I = \tfrac{1}{4} U \tag{4.11}$$

A useful corollary of (4.8) is Wien's displacement law, that the frequency carrying the maximum energy density is proportional to the temperature, and analogously for the wavelength

$$\nu_W = \frac{2.821}{2\pi} T \qquad \lambda_W = \frac{2\pi}{4.965} T^{-1} \tag{4.12}$$

PROBLEM 4.4. The microwave background has $T = 2.725\,\mathrm{K}$. Calculate ν_W in Hz, and the photon density n_γ in photons per cm^3.

PROBLEM 4.5. The solar constant is measured as $\simeq 1350\,\mathrm{W/m^2}$. Estimate its value theoretically, considering the solar surface as a blackbody radiator with $\lambda_W = .5\,\mu\mathrm{m}$, and assuming that the Earth receives 2×10^{-5} of what the sun emits.

5. The Chandrasekhar mass scale

By the end of the 19th century, the macroscopic properties of the Sun (mass, density, luminosity, surface temperature) were fairly well measured, and starting to be measured for nearby stars as well, and some theory about the inside of a star was starting to be developed. What was not yet understood was the interaction between gravity and microphysical processes, which is crucial to a star's working.

But once quantum mechanics was developed, the basic theory of stellar structure followed quite quickly, around the 1930s. The contributions of Chandrasekhar are especially important. Stellar structure involves, unfortunately, some unpleasantly nonlinear equations, and getting accurate quantitative results is beyond what we can do in this course. We can, however, gain a lot of insight from some extremely rough calculations.

In this chapter we will study the pressure inside a star, which will lead us to the mass scale for stars. One strange-looking number[6]

$$N_L \equiv m_N^{-3} \tag{5.1}$$

will play a crucial role. The number of nucleons in a star is of order N_L. In particular, the Sun has

$$N_\odot = 0.54\, N_L \tag{5.2}$$

The famous Chandrasekhar mass has

$$N_{\rm ch} = 0.78\, N_L \tag{5.3}$$

§5.1 Hydrostatic equilibrium. Consider a shell at radius r inside a star. We write $n_N(r)$ for the number density of nucleons there. The number of nucleons $N(r)$ enclosed by this shell is clearly given by

$$\frac{dN(r)}{dr} = 4\pi r^2 n_N(r) \tag{5.4}$$

The local mass density $n_N(r)\, m_N$ (assuming nucleons dominate the mass). Hence the inward pressure is given by

$$\frac{dP(r)}{dr} = -m_N^2 \frac{N(r) n_N(r)}{r^2} \tag{5.5}$$

The equations (5.4: mass continuity) and (5.5: hydrostatic equilibrium) are two of the four differential equations of stellar structure. (We will come to the other two later.)

The inward pressure $P(r)$ will tend to make a star contract. For equilibrium, there must be an equal and opposite resisting pressure. This depends on the composition. For

[6] Which following Brandon Carter we will call the Landau number

example, a classical ideal gas has $P = n_N T$. In general, we denote it by an equation of state

$$n_N(P, T) \tag{5.6}$$

§5.2 ROCK AND ICE. Solid bodies are held up by electrostatic repulsion between electrons in atoms and molecules. Atoms have a radius $\sim (\alpha m_e)^{-1}$, but usually molecules are not as tightly packed as this, because bonds are anisotropic. The approximate formula

$$n_N \simeq A \left(\frac{\alpha m_e}{4}\right)^3 \tag{5.7}$$

where A is the mean mass number per molecule ($A = 6$ for ice, $A \simeq 30$ for rock) works quite well. In this rough equation of state, we are ignoring the temperature dependence.

PROBLEM 5.1. Verify that (5.7) does give approximately the density of ice.

PROBLEM 5.2. Estimate the pressure at the centre of a rock/ice planet in terms of n_N and N.

For the Earth $A \simeq 30$ and $N/N_L =\simeq 1.5 \times 10^{-6}$. What is then the pressure at the centre of the Earth in SI units?

§5.3 THE VIRIAL THEOREM. Let us multiply the equation of hydrostatic equilibrium (5.5) by $4\pi r^3$ and integrate over r

$$\int 4\pi r^3 \frac{dP(r)}{dr}\, dr = -m_N^2 \int_0^R \frac{N(r) n_N(r)}{r} 4\pi r^2\, dr \tag{5.8}$$

The left hand side, on integrating by parts, becomes $-3PV$. The right hand side is just the gravitational binding energy. Hence we have

$$PV = \tfrac{1}{3} E_{\text{grav}} \tag{5.9}$$

which is known as the virial theorem.

For an ideal gas $T \propto P/n_N$ hence contraction makes it hotter.

For a sphere of radius R with constant n_N, we have $N(r) = Nr^3/R^3$ which gives

$$E_{\text{grav}} = m_N^2 N n_N \int_0^R \frac{4\pi r^4}{R^3}\, dr = \frac{4\pi R^2}{5} m_N^2 N n_N = \frac{3 m_N^2 N n_N V}{5R}$$

and hence

$$P = \frac{m_N^2 N n_N}{5R}$$

Now we substitute

$$R = \left(3/(4\pi)\, N/n_N\right)^{1/3} \simeq \tfrac{3}{5} (N/n_N)^{1/3} \tag{5.10}$$

we have

$$P \simeq \tfrac{1}{3} n_N^{4/3} \left(\frac{N}{N_L}\right)^{2/3} \tag{5.11}$$

This approximate form of the virial theorem will prove very useful.

§5.4 THE CLAYTON MODEL. Pressure-density solutions appropriate for stars usually require numerical solution. The Clayton model is one of the few that can be solved without a computer but still is reasonably realistic.

We assume we know N and n_N (and hence R) for some star. We model the density distribution as having a dense core of radius

$$a = R/X \tag{5.12}$$

where X is a parameter in the model, and a less-dense outer part. We then consider the following model form

$$\frac{dP}{dr} = -\frac{4\pi}{3} m_N^2\, n_0^2\, r\, e^{-r^2/a^2} \tag{5.13}$$

Here n_0 depends (in a way to be determined) on N, n_N, Comparing with the hydrostatic equilibrium equation (5.5) we have

$$\frac{N(r) n_N(r)}{r^2} = \frac{4\pi}{3} n_0^2 r\, e^{-r^2/a^2} \tag{5.14}$$

Multiplying this equation by

$$dN = 4\pi n_N(r) r^2\, dr$$

and rearranging, we get

$$N^2(r) = \frac{32\pi^2}{3} n_0^2 \int_0^r r^5 e^{-r^2/a^2}\, dr$$

and working out the integral we derive

$$N(r) = \frac{4\pi}{3} n_0 a^3 \Phi(x) \qquad x = r/a \tag{5.15}$$

where

$$\Phi(x) = \sqrt{6 - 3e^{-x^2}(x^4 + 2x^2 + 2)} = \begin{cases} x^3 & \text{if } x \ll 1 \\ \sqrt{6} & \text{if } x \gg 1 \end{cases} \tag{5.16}$$

From (5.14) we can now derive

$$n_N(r) = n_0 \frac{x^3 e^{-x^2}}{\Phi(x)} \tag{5.17}$$

whereas (5.13) directly gives

$$P(r) = \frac{2\pi}{3} m_N^2 n_0^2 a^2 \left(e^{-x^2} - e^{-X^2}\right) \tag{5.18}$$

To get an explicit expression for n_0, we put the requirement

$$N(R) = N$$

in (5.15), which gives

$$n_0 \simeq \frac{X^3 n_N}{\sqrt{6}} \tag{5.19}$$

A Clayton model with $X \simeq 5$ is a reasonable model for the solar interior.

§5.5 DEGENERACY PRESSURE. A degenerate Fermi gas is the large μ limit of (4.4). Here we are principally interested in electrons, for which $g = 2$, in which case the density of states becomes

$$n(p) = \frac{p^2}{\pi^2} \tag{5.20}$$

and the levels up to the Fermi momentum p_F fill.

The number density is

$$n = \frac{{p_F}^3}{3\pi^2} \tag{5.21}$$

The pressure is

$$P = \frac{1}{\pi^2} \int_0^{p_F} \frac{pv}{3} p^2 \, dp \qquad p = \frac{mv}{\sqrt{1 - v^2}} \tag{5.22}$$

Putting in the cases for $v(p)$, integrating and simplifying, we get

$$P \simeq 2m^{-1} n^{5/3} \quad \text{non-relativistic} \tag{5.23}$$

and

$$P \simeq \tfrac{3}{4} n^{4/3} \quad \text{ultra-relativistic} \tag{5.24}$$

If we take $p_F \simeq m$ then (5.21) gives

$$n \simeq \frac{m^3}{30} \quad \text{relativistic threshold} \tag{5.25}$$

PROBLEM 5.3. Derive a general expression for the degeneracy pressure (that is, not in the non-relativistic or ultra-relativistic limits) as a function of $u = p_F/m$.

§5.6 PROTO-STARS. For large-enough masses, the inward pressure (5.5) exceeds what can be supported by electrostatic pressure. If fusion occurs, the pressure of hot gas can hold up the star. But if not, degeneracy pressure will provide support, with n_N increasing drastically if necessary.

For a hydrogen proto-star $n_e = n_N$. Using this and equating the virial pressure (5.11) and the degeneracy pressure (5.23), we get

$$n_N \simeq \frac{m_e{}^3}{200} \left(\frac{N}{N_L} \right)^2 \tag{5.26}$$

Comparing with (5.21) we have

$$p_F \simeq \frac{m_e}{2} \left(\frac{N}{N_L} \right)^{2/3}$$

and hence the most energetic particles will have energy

$$E_{\max} \simeq \frac{m_e}{8} \left(\frac{N}{N_L} \right)^{4/3} \tag{5.27}$$

which eventually results in fusion.

PROBLEM 5.4. Consider a rock/ice planet with $A = 1$ (frozen hydrogen). Around what N will the centre become degenerate?

§5.7 STELLAR REMNANTS. A stellar remnant is mostly helium, $n_e = \frac{1}{2} n_N$. Equating the pressures in (5.11) and (5.23) in this case we get

$$n_N \simeq \frac{m_e{}^3}{7} \left(\frac{N}{N_L} \right)^2 \tag{5.28}$$

But this easily becomes relativistic, as is evident from comparing with (5.25). In the ultra-relativistic case, we use the pressure (5.24) instead, which gives

$$\left(\frac{N}{N_L} \right) \simeq \frac{5}{6} \tag{5.29}$$

or Chandrasekhar's limit.

Beyond, we have neutron degeneracy pressure

$$n_N \simeq \frac{m_N^3}{200} \left(\frac{N}{N_L} \right)^2 \tag{5.30}$$

This is a regime of neutron stars.

PROBLEM 5.5. A neutron star never gets a chance to cross the threshold (5.25). Before that can happen, the radius crosses below the Schwarzschild radius $2Nm_N$ and the star becomes a black hole. Estimate the N/N_L for which this will happen.

6. Nuclear fusion

From the early days of nuclear physics, it was evident that nuclear fusion of Hydrogen to Helium was the energy source in stars.

The problem was: how do you get two nucleii close enough, despite their strong electrostatic repulsion, that the even stronger but very short-range nuclear force takes over and makes the fusion reaction happen. One way is to raise the temperature enough that kinetic energy overcomes electrostatic repulsion. That is what hydrogen bombs and (in more controlled fashion) tokamaks do. But stars do something more subtle. As Gamow first realized, and then Bethe worked out in detail already in the 1930s,[7] in stars the electrostatic repulsion is evaded by quantum tunnelling. This is a very low probability event, but stars can afford that.

Let us now study how quantum tunnelling allows fusion in stars at 'warm' temperatures.

§6.1 THE REACTIONS. There are several possible fusion reaction chains from H to He. Here is one

$$\begin{aligned} p + p &\to {}^2\mathrm{H} + e^+ + \nu_e + \gamma \\ {}^2\mathrm{H} + p &\to {}^3\mathrm{He} + \gamma \\ {}^3\mathrm{He} + {}^3\mathrm{He} &\to {}^4\mathrm{He} + 2p + \gamma \end{aligned} \tag{6.1}$$

The mass of ^{4}He is $\sim 1\%$ less than the mass of four protons, and this gets liberated as radiation.

The reaction set (6.1) and its variants are known as the pp chain. Another important set of reactions is the CNO cycle; the heavier nucleii acts as catalysts in the reaction, but the end result is still fusion of H to He.

§6.2 FORMULATING THE REACTION RATE. Reaction rates between two species (A and B say) are generically of the form

$$n_A n_B \langle \sigma v \rangle \tag{6.2}$$

The $n_A n_B$ factor gives the rate of encounters, while $\langle \sigma v \rangle$ is proportional to the probability of an encounter leading to a reaction.

$$\langle \sigma v \rangle = \int \sigma v \, P(v) \, dv \tag{6.3}$$

where

$$P(v) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-E/T\right) \, 4\pi v^2 \qquad E = \tfrac{1}{2} m v^2 \tag{6.4}$$

is the velocity distribution in a Maxwell-Boltzmann gas. Note that $P(v) \, dv = 1$.

Classically, for the nuclei to come within the range of the nuclear force (call it r_0), they would need $E \simeq \alpha / r_0$. Now, r_0 is (coincidentally) $\sim \alpha / m_e$, hence $E \sim m_e$ would

[7] See http://nobelprize.org/nobel_prizes/physics/articles/fusion/ for an interesting article on the history.

be required. From (5.27) we see that proto-stars do not reach this even for the most energetic particles for $N \simeq N_L$ and yet stars even with $N \simeq 0.04 N_L$ manage to start fusion. Something else must be happening.

§6.3 THE GAMOW FACTOR. Recall the Schrödinger equation

$$-\frac{1}{2m}\nabla^2\psi = (E - V)\psi \tag{6.5}$$

The relevant m here is the reduced mass. Assuming spherical symmetry (6.5) becomes

$$\frac{1}{r}\frac{d}{dr^2}(r\psi) = 2m(V - E)\psi \tag{6.6}$$

If V is constant, the solution is

$$r\psi \propto \exp\left(\pm\sqrt{2m(V-E)}\, r\right) \tag{6.7}$$

Since volume element $\propto r^2$, the tunnelling probability through a length Δr is

$$\exp\left(-2\sqrt{2m(V-E)}\,\Delta r\right) \tag{6.8}$$

We exclude the other sign on physical grounds.

Now we put $V = \alpha/r$ and integrate to get the tunnelling probability

$$\exp\left(-2\sqrt{2mE}\int\sqrt{\frac{\alpha}{Er} - 1}\, dr\right) \tag{6.9}$$

We take the integral over the full interval that argument of the $\sqrt{}$ is positive:

$$\int = \frac{\pi\alpha}{2E} \tag{6.10}$$

This gives the Gamow factor

$$\exp\left(-\sqrt{E_G/E}\right) \tag{6.11}$$

where

$$E_G = 2m(\pi\alpha)^2 \tag{6.12}$$

PROBLEM 6.1. Consider an interaction between two hydrogen atoms. In the atom, the Coulomb barrier is shielded outside a radius $r_1 = 1/(m_e\alpha)$. In effect, the integral (6.9) is $\int_0^{r_1}$ instead of $\int_0^{\alpha/E}$. Show that if

$$E/m_e \ll \alpha^2$$

the Gamow factor is replaced by

$$-2^{\frac{5}{2}}\sqrt{m/m_e}$$

This applies in muon-catalyzed fusion.

§6.4 Reaction cross section. The previous section shows that σ will have a Gamow factor. The rest of σ requires nuclear theory and/or accelerator experiments to evaluate, but the form

$$\sigma = \frac{S_0}{E}\exp\left(-\sqrt{E_G/E}\right) \tag{6.13}$$

is found to be good approximation.

Now returning to (6.3) and putting

$$E = \tfrac{1}{2}mv^2 \Rightarrow \frac{v^3\,dv}{E} = \frac{2\,dE}{m^2}$$

we get

$$\langle\sigma v\rangle = (m\pi)^{-\frac{1}{2}}(2/T)^{\frac{3}{2}}S_0\int_0^\infty \exp\left(-E/T - \sqrt{E_G/E}\right)\,dE \tag{6.14}$$

§6.5 Integral over energy. We need to evaluate

$$\int_0^\infty \exp\left(-E/T - \sqrt{E_G/E}\right)\,dE \tag{6.15}$$

Let us denote the integrand by $f(E)$. We approximate the integral by

$$f(E_0)\int_0^\infty \exp\left(-\frac{(E-E_0)^2}{2\Delta^2}\right)\,dE \tag{6.16}$$

in which case the integral is

$$\sqrt{2\pi}\Delta f(E_0) \tag{6.17}$$

Introduce

$$\theta^3 \equiv \frac{E_G}{4T} \tag{6.18}$$

which gives

$$f(E) = \exp\left(-E/T - 2\theta^{\frac{3}{2}}\,(T/E)^{\frac{1}{2}}\right) \tag{6.19}$$

From this we easily derive

$$E_0 = \theta T \tag{6.20}$$

and

$$f(E_0) = \exp(-3\theta) \tag{6.21}$$

Then we have

$$\Delta^2 = -\frac{f(E_0)}{f''(E_0)} = \tfrac{2}{3}\theta T^2 \tag{6.22}$$

The integral is

$$\sqrt{\tfrac{4}{3}\pi\theta}\, T \exp(-3\theta) \tag{6.23}$$

§6.6 TEMPERATURE DEPENDENCE. From (6.23) and (6.14) we see that

$$\langle \sigma v \rangle \propto T^{-2/3} \exp(-3\theta) \tag{6.24}$$

The temperature sensitivity is still strong, but much less vicious that $\exp(-E/T)$. It is usual to define an effective exponent

$$\langle \sigma v \rangle \propto T^{\theta - \frac{2}{3}} \tag{6.25}$$

This characterises the reaction rate as a power of T with index depending on E_G.

PROBLEM 6.2. Order the reactions in (6.1) by temperature dependence.

7. The main sequence of stars

The energy that stars generate from nuclear fusion is not radiated away immediately, because stars are quite opaque. The opacity of a star is what keeps it regulated and shining for a long time. We now see how.

In this chapter our expressions will be even more approximate than when we were studying stellar mass scales. This is because some of the previously already approximate factors enter in the third or fourth power. When this happens we will use $\sim$ rather than $\simeq$.

§7.1 LUMINOSITY AND RADIATIVE TRANSFER. We now meet the other two equations of stellar structure.

One is the energy conservation equation relates power output $\epsilon(r)$ and luminosity $L(r)$.

$$\frac{dL(r)}{dr} = 4\pi r^2 \epsilon(r) \tag{7.1}$$

More subtle is the equation of radiative transfer. We can think of it as kind of like the hydrostatic pressure equation, but for radiation pressure.

$$\frac{dP_\gamma(r)}{dr} = -\frac{\kappa n_N L(r)}{4\pi r^2} \tag{7.2}$$

Here κ is the opacity per particle.

§7.2 PRESSURE RATIOS. Recall that

$$P_{\text{gas}} = n_N T \qquad P_\gamma = \frac{\pi^2}{45} T^4 \tag{7.3}$$

which gives us

$$\frac{P_{\text{gas}}}{P_\gamma} \simeq 5 n_N T^{-3} \tag{7.4}$$

In a regime where gas-pressure dominates, we put $T = P/n_N$ in (7.4) which gives

$$\frac{P_{\text{gas}}}{P_\gamma} \sim n_N (P/n_N)^{-3}$$

and then substituting for P from the virial theorem (5.11) gives

$$\frac{P_{\text{gas}}}{P_\gamma} \sim \left(\frac{N}{N_L}\right)^{-2} \qquad \text{if } P_{\text{gas}} \gg P_\gamma \tag{7.5}$$

§7.3 OPACITY. Accurate calculation of opacity is in practice the most complicated part of stellar structure. Lifetimes of work have been devoted to it. Fortunately though, there are some simple approximate formulas.

Thomson opacity is

$$\kappa_{\mathrm{T}} = \frac{8\pi}{3}\left(\frac{\alpha}{m_e}\right)^2 \tag{7.6}$$

Kramers opacity has two components. Free-free scattering gives

$$\kappa_{\mathrm{ff}} \simeq \tfrac{1}{4}\kappa_{\mathrm{T}}\left(\frac{m_e}{T}\right)^{1/2} \alpha n_{\scriptscriptstyle N} T^{-3} \tag{7.7}$$

Bound-free has similar dependence.

Overall, we can write Kramers opacity as

$$\kappa_{\mathrm{K}} \sim \kappa_{\mathrm{T}}\alpha\left(\frac{m_e}{T}\right)^{1/2} \frac{P_{\mathrm{gas}}}{P_\gamma} \tag{7.8}$$

PROBLEM 7.1. Another consequence of scattering is that inside a star, a photon travels only a short distance before being scattered by an electron. The typical "mean free path" between scatterings is

$$l \sim \frac{1}{\kappa_{\mathrm{T}} n_e}$$

Hence photons in a star effectively travel in a random walk of step-length l. After s steps of such a random walk, the typical linear distance travelled is $\sqrt{s}\,l$.

Assuming a star has $N \simeq N_L$ and approximately the density of water, show that the time scale for a photon to travel from the centre of a star to the surface is

$$\frac{\alpha^3}{m_e m_{\scriptscriptstyle N}^2}$$

§7.4 LUMINOSITY. We have

$$L \sim \frac{R P_\gamma}{\kappa n_{\scriptscriptstyle N}}$$

We can eliminate R using

$$R \sim n_{\scriptscriptstyle N}^{-1/3} m_{\scriptscriptstyle N}^{-1}\left(\frac{N}{N_L}\right)^{1/3} \tag{7.9}$$

gives

$$L \sim \frac{P_\gamma}{m_{\scriptscriptstyle N}\kappa} n_{\scriptscriptstyle N}^{-4/3}\left(\frac{N}{N_L}\right)^{1/3} \tag{7.10}$$

If P_γ dominates we eliminate it using the virial theorem (5.11)

$$L \sim \frac{1}{m_N \kappa_T} \left(\frac{N}{N_L} \right) \tag{7.11}$$

If gas pressure and Thomson opacity dominate, we use (7.5) in (7.10) and then (5.11)

$$L \sim \frac{1}{m_N \kappa_T} \left(\frac{N}{N_L} \right)^3 \tag{7.12}$$

If gas pressure and Kramers opacity dominate, then (7.8) gives us an extra factor, hence

$$L \sim \frac{1}{m_N \kappa_T} \alpha^{-1} \left(\frac{T}{m_e} \right)^{1/2} \left(\frac{N}{N_L} \right)^5 \tag{7.13}$$

Thus, the luminosity scale is set by

$$\frac{{m_e}^2}{\alpha^2 m_N} \tag{7.14}$$

The age scale is set by

$$\eta \frac{\alpha^2}{{m_e}^2 m_N} \tag{7.15}$$

where η is the nuclear binding-energy fraction, about 10^{-2}.

§7.5 Effective temperature. Defined by

$$L \sim R^2 T_{\rm eff}^4 \tag{7.16}$$

Substituting for R from (7.9) and then eliminating n_N in favour of P using the virial theorem (5.11) yet again, we have

$$T_{\rm eff}^4 \sim L P^{\frac{1}{2}} m_N^2 \left(\frac{N}{N_L} \right)^{-1}$$

If radiation pressure dominates, then from (7.11) we have

$$T_{\rm eff}^4 \sim \frac{m_N}{\kappa_T} T^2 \tag{7.17}$$

If gas pressure and Thomson opacity dominate, we use (7.12) for the luminosity and the pressure factor (7.5)

$$T_{\rm eff}^4 \sim \frac{m_N}{\kappa_T} T^2 \left(\frac{N}{N_L} \right) \tag{7.18}$$

and if Kramers opacity dominates we pick up an extra factor from (7.8)

$$T_{\rm eff}^4 \sim \frac{m_N}{\kappa_T} \alpha^{-1} \left(\frac{T}{m_e} \right)^{1/2} T^2 \left(\frac{N}{N_L} \right)^3 \tag{7.19}$$

Considering massive stars again, we have

$$T_{\text{eff}}^4 \sim \frac{{m_e}^2 m_{\text{N}}}{\alpha^2} T^2 \tag{7.20}$$

Hence $T_{\text{eff}} \ll T$.

PROBLEM 7.2. We can infer T_{eff} using the observation that $\lambda_W = .5\,\mu\text{m}$. We also know that the Sun has $N \simeq 0.5 N_L$. Using these values, compute what equation (7.18) gives for T.

Look up the solar interior temperature and comment.

8. The expanding universe

In the last part of this course, we will study the basics of cosmology. The modern theory of cosmology actually started out a little earlier than the theory of stellar structure, but as the key observations were very much harder, its development took much longer. Debates about fundamental issues are still going on.

When Einstein developed general relativity, it was natural to ask: what did this new theory of gravity imply on the very largest scales? Working this out involved some trial and error, but eventually everyone agreed that Friedmann got it right in 1922. Friedmann showed that if one assumes that on the largest scales the universe has no special locations or directions (the so-called cosmological principle), Einstein's field equations simplify beautifully into what we now call the Friedmann equation, and the universe cannot be static, it must be expanding or contracting.

§8.1 THE FRIEDMANN EQUATION. The key variable is the scale-factor of the universe $a(t)$. The Friedmann equation (or equations—sometimes it is expressed as two equations) is a differential equation for $a(t)$, depending on the density of the universe.

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi}{3}\rho \\ \rho &= \frac{\rho_m}{a^3} + \frac{\rho_\gamma}{a^4} + \Lambda + \frac{K}{a^2} \end{aligned} \tag{8.1}$$

The scale factor is dimensionless, and by convention $a(t) = 1$. One often rescales quantities (lengths, densities) from past times to $a = 1$. Such rescaled quantities are said to be 'comoving'.

The density ρ is not just matter, it can have several components, of which four are thought to be important. The second line in (8.1) is a sort of equation of state for ρ, in terms of four constants $\rho_m, \rho_\gamma, \Lambda, K$.

(i) ρ_m is the comoving density of matter, and the actual density $\propto a^{-3}$ as expected.

(ii) ρ_γ is the comoving density of photons. The actual photon density (averaged over cosmological scales) is $\propto a^{-3}$, but the wavelengths are $\propto a$, so the radiation density $\propto a^{-4}$.

(iii) Λ is a very strange thing, which we call 'the cosmological constant' to hide the fact that we have no idea where it comes from.

(iv) K (the constant $k = -\frac{8\pi}{3}K$ is often used instead) denotes the cosmologically averaged curvature of space. Though it is not a density, its dynamical effects are somewhat similar.

§8.2 DIMENSIONLESS DENSITIES. Instead of the four constants in (8.1), it is often convenient to write the Friedmann equation in terms of four dimensionless constants (the Ω constants) that add up to 1, and an overall scale H_0, thus:

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_\gamma}{a^4} + \Omega_\Lambda + \frac{\Omega_K}{a^2} \right) \tag{8.2}$$

where

$$\Omega_m \equiv \frac{\rho_m}{\rho_m + \rho_\gamma + \Lambda + K} \tag{8.3}$$

and so on. H_0 is known as Hubble's constant and H_0^{-1} as the Hubble time. The current best values are

$$H_0^{-1} \simeq 13.7\,\mathrm{Gyr} \qquad \Omega_m \simeq 0.25 \qquad \Omega_\Lambda \simeq 0.75 \qquad \Omega_K \simeq 0 \tag{8.4}$$

while Ω_γ is nonzero but negligible for most purposes. For historical reasons, H_0 is usually expressed in weird units

$$H_0 \simeq 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{8.5}$$

Two cases that can be solved exactly are of special interest, even though the parameter values are now considered unrealistic. The first is the Einstein-de Sitter universe, which has $\Omega_m = 1$ and all others zero. The solution $a = t^{\frac{2}{3}}$ is easily derived. The second interesting special case is the dust universe, which has $\Omega_K = 1 - \Omega_m$ and all the others zero.

§8.3 Interpretation. The scale factor is directly observable, as the redshift of the wavelength of spectral lines.

Redshift is conventionally defined as z where $1+z = 1/a$. Traditionally, z was multiplied by the speed of light and expressed as a recession velocity, but not so much nowadays, when many objects have been measured with $z > 1$.

A Newtonian interpretation of the Friedmann equation is also possible, at least for a dust universe. Introducing $M = \frac{4\pi}{3} a^3 \rho$ and $k = -\frac{8\pi}{3} K$ we can write (8.1) as

$$\tfrac{1}{2}\dot{a}^2 - \frac{M}{a} = -\tfrac{1}{2}k \tag{8.6}$$

This has the same form as the energy equation for a sphere of radius a with enclosed mass M. We can think of this sphere as having exploded from a point, and now self-gravitating. If $k > 0$ (or $\Omega_K > 1$) it will collapse again, otherwise not.

Problem 8.1. We already have the solution for the Newtonian radial collapse of a spherical shell. This is equation (1.20) with $e = 0$ (but beware the completely different meaning of a). Using this known solution as a guide, solve for $a(t)$ in a dust universe with $\Omega_m = \Omega > 1$. The trick is to reduce

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega}{a} - (\Omega - 1) \right)$$

to the trigonometric identity

$$\cot^2 u = \csc^2 u - 1$$

PROBLEM 8.2. The Hubble time is very far from any microphysical time scale. Nevertheless, we can give H_0 some kind of microphysical interpretation, by considering the condition for the universe to be transparent.

Taking H_0 as of order the size of the observable universe, and $1/(\kappa_T n_e)$ as a mean free path, the transparency condition is

$$\frac{H_0}{\kappa_T n_e} > 1$$

Now, roughly a fifth of the matter in the universe is hydrogen: that is, $m_N n_N \simeq \frac{1}{5}\rho_m$. Using this information, show that the transparency condition becomes

$$H_0 < 20 \frac{{m_e}^2 m_N}{\alpha^2}$$

§8.4 THE METRIC. As is well known, general relativity implies curved spacetimes. Curvature of spacetime enters through $a(t)$. In addition, at constant t, space on its own may (but need not) be curved. Space curvature (which from the cosmological principle must be constant) is specified by Ω_K.

If $\Omega_K < 0$ (which just to be confusing is called positive curvature) space is spherical, or rather a four-dimensional sphere embedded in three dimensions. It is as if we were two-dimensional beings living on the surface of a ball.

If $\Omega_K > 0$ (negative curvature) then space is hyperbolic. This case is more difficult to get intuition for, but has been explored in art.[8]

One way to specify a constant space curvature mathematically is to define a 'circumferential comoving radius'

$$\tilde{r} = \begin{cases} r & \text{if } \Omega_K = 0 \\ \sin(\kappa r)/\kappa & \Omega_K < 0 \\ \sinh(\kappa r)/\kappa & \Omega_K > 0 \end{cases} \qquad \kappa \equiv H_0 |\Omega_K|^{\frac{1}{2}} \tag{8.7}$$

Lengths are then expressed through a metric

$$ds^2 = a^2(t)\Big(dr^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\Big) - dt^2 \tag{8.8}$$

This is known as the Robertson-Walker metric in comoving form. The overall name for the cosmology is FLRW for Friedmann-Lemaitre-Robertson-Walker.

By the way, we will see below $r \propto H_0^{-1}$ and hence the argument of sin and sinh in (8.7) does not really contain H_0.

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[8] Most famously by M.C. Escher in four pictures called *Circle Limit I–IV*. For a beautiful Escher-inspired gallery of hyperbolic images, look up 'Hyperbolic Escher' by Jos Leys.

§8.5 DISTANCES AND TIMES. In a universe which is expanding while light travels across it, with space itself possible curved, the whole concept of distance becomes more complicated. The following integrated quantities are important.

First we have the lookback time, expressed in terms of the a of an object

$$t = \int_a^1 \frac{da}{\dot{a}} \tag{8.9}$$

When a is small, comoving lengths are smaller physically. Hence light gets through more comoving length when a is small. The comoving distance is what light traverses in the lookback time.

$$r = \int_a^1 \frac{da}{\dot{a}a} \tag{8.10}$$

The angular-diameter distance

$$d_{\text{ang}} = \tilde{r}a \tag{8.11}$$

relates the apparent size of an object to its physical size.

As the universe expands, a telescope of given physical aperture subtends a small angle at the source, hence it picks up less light. This leads to the definition of the luminosity distance.

$$d_{\text{lum}} = \tilde{r}/a \tag{8.12}$$

PROBLEM 8.3. Find the a at which, in the Einstein-de Sitter case, d_{ang} has its maximum value.

PROBLEM 8.4. An object at comoving distance r is moving with non-relativistic speed v transverse to the line of sight. What is its angular velocity on the observer's sky?

§8.6 REDSHIFT DRIFT. So far, we have tacitly assumed that all observations happen at one time. What if we observe over $a(t)$ period of time? Can we see detect the universe expanding directly, rather than inferring it indirectly?

When we take t_{obs} into account, the measured quantity is really

$$1 + z = \frac{a(t_{\text{obs}})}{a(t)} \tag{8.13}$$

Here t means t at the source. Consider

$$\Delta z = \frac{a(t_{\text{obs}} + \Delta t_{\text{obs}})}{a(t + \Delta t)} - \frac{a(t_{\text{obs}})}{a(t)}$$

Substitute $\Delta t = (a/a_{\text{obs}})\Delta t_{\text{obs}}$ and simplify. The redshift drift is

$$\frac{dz}{dt_{\text{obs}}} = \frac{\dot{a}_{\text{obs}} - \dot{a}}{a} \tag{8.14}$$

Since $\dot{a}$ is of order $H_0 \sim 10^{-10}\,\text{yr}^{-1}$ the accuracy required is quite demanding. Still, modern spectrographs can achieve accuracies of 10^{-9} under optimal conditions, so there are plans to try and measure the redshift drift over a period of several years.

9. The microwave background

The Friedmann equations give the consequences of gravity on cosmological scales. It was then Gamow (a past student of Friedmann) who in the 1940s realized that microphysics had important consequences too. An expanding universe would gradually cool, with different physical effects happening at different $a(t)$. By calculating the synthesis of Helium from Hydrogen in the very early universe (the fascinating topic of Big-Bang nucleosynthesis, which we will have to leave for a later course!) Gamow made a bold prediction that the universe would have a temperature around 5 K now.

In fact, a temperature measurement of about 2.3 K for interstellar gas had already been made by McKellar. But nobody made the connection until the 1960s, when Penzias and Wilson detected a mysterious microwave signal coming from all directions, and the pieces of the puzzle fell into place.

In 2009, detailed study of the cosmic microwave background (CMB) is one of the most active areas of astrophysics. Let us try and get a feeling for why.

§9.1 Radiation and matter densities. The microwave background is a photon gas. In fact, it is closer to the theoretical ideal photon gas than anything made in a lab. The temperature has been measured quite accurately as

$$T_0 = 2.725 \text{ K} \tag{9.1}$$

and this is the current average temperature of the universe. The radiation density in (8.1) is therefore

$$\rho_\gamma = \frac{\pi^2}{15} T_0^4 \tag{9.2}$$

Since $T \propto 1/a$ we can write the number density of nucleons as

$$n_N = \bar{\eta} T^3 \tag{9.3}$$

and hence the matter density is

$$\rho_m = \bar{\eta} T_0^3 m_N (1 + x_{\text{dark}}) \tag{9.4}$$

Here x_{dark} is the ratio of dark to ordinary matter. The numbers

$$\bar{\eta} \simeq 1 \times 10^{-10} \qquad x_{\text{dark}} \simeq 4 \tag{9.5}$$

are much more approximate than T_0.

The notation $\bar{\eta}$ is non-standard, conventionally $\eta \equiv n_N/n_\gamma$ is used. It is easy to relate the two quantities. Since $n_\gamma = 2\zeta(3)/\pi^2\, T^3$ it follows from (9.3) that $\bar{\eta}/\eta = 2\zeta(3)/\pi^2$.

From (8.1) we see that matter and radiation densities will be equal at $a_{\text{eq}} = \rho_\gamma/\rho_m$. Thus

$$a_{\text{eq}} = \frac{\pi^2}{15(1 + x_{\text{dark}})} \frac{T_0}{\bar{\eta} m_N} \simeq \frac{1}{600(1 + x_{\text{dark}})} \tag{9.6}$$

PROBLEM 9.1. The sun has a radius of about 2 light-seconds and a surface temperature around 2×10^3 that of the CMB. How far from the sun does the photon-number flux from the sun approximately equal that of the CMB? How far away do the energy fluxes become approximately equal?

§9.2 RECOMBINATION. A photon gas is generated through repeated interaction with matter. For CMB photons the last interaction happened at quite small a. This was when the universe cooled enough that an electron-proton plasma, which interacts strongly with photons, recombined into atomic hydrogen, which is transparent.

To calculate when this happened, we consider the equilibrium between three Maxwell-Boltzmann gases of the type (4.7) with n_e, n_p, n_H. Putting

$$\mu_H - \mu_p - \mu_e = Q = \tfrac{1}{2} m_e \alpha^2$$

and defining the ionization fraction

$$x = \frac{n_e}{n_N} \tag{9.7}$$

where n_N is the total density of protons whether in atoms or plasma, we have

$$\frac{1-x}{x^2} = \frac{n_N n_H}{n_e n_p} = \bar{\eta} \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{Q/T} \tag{9.8}$$

Large T give $x \to 1$ (completely ionized), but declines to $x \simeq 0.5\%$ at 3000 K. Thus

$$a_{\rm cmb}^{-1} \simeq 1100 \tag{9.9}$$

§9.3 HORIZONS. The horizon at any $a(t)$ is the distance light would have travelled since the Big Bang ($t = 0$). At the CMB we have

$$\Delta_{\rm hor} = a_{\rm cmb} \int_0^{a_{\rm cmb}} \frac{da}{\dot{a}a} \tag{9.10}$$

The integral gives the comoving size, and we multiply by $a_{\rm cmb}$ to get the physical size.

To estimate $\Delta_{\rm hor}$ recall from (8.2) that

$$\dot{a}a = H_0 \left(\Omega_\gamma + \Omega_m a + \Omega_K a^2 + \Omega_\Lambda a^4\right)^{\frac{1}{2}}$$

In the range

$$a_{\rm eq} \ll a \ll 1$$

we have

$$\dot{a}a \simeq H_0 (\Omega_m a)^{\frac{1}{2}}$$

Thus we have

$$\Delta_{\text{hor}} \simeq \frac{2a_{\text{cmb}}^{\frac{3}{2}}}{H_0 \Omega_m^{1/2}} \tag{9.11}$$

To work out how large this is on the sky, we need $d_{\text{ang}}(a_{\text{cmb}})$. This is easy to calculate only in the Einstein-de Sitter case, for which we have

$$d_{\text{ang}} = a_{\text{cmb}} \int_{a_{\text{cmb}}}^{1} \frac{da}{\dot{a}a} \simeq \frac{2a_{\text{cmb}}}{H_0 \Omega_m^{1/2}} \tag{9.12}$$

Hence the angular horizon size is

$$\theta_{\text{hor}} \simeq a_{\text{cmb}}^{\frac{1}{2}} \tag{9.13}$$

for Einstein-de Sitter, and of the same order for the values (8.4).

Analogous to the horizon, there is also the sound horizon, which is the maximum distance a pressure wave in the photon-matter plasma can travel. Comparing (9.6) and (9.9) we can see that while the overall density is more than the radiation density, the latter is still more than the density in ordinary matter. Dark matter does not carry the sound waves, since it interacts only gravitationally. Hence photons still dominate the acoustic waves up to a_{cmb}. The speed of sound in a photon-dominated plasma is $1/\sqrt{3}$ of the speed of light. Hence

$$\theta_{\text{hor}}^{\text{sound}} \simeq \left(\tfrac{1}{3} a_{\text{cmb}}\right)^{\frac{1}{2}} \simeq 1^\circ \tag{9.14}$$

Fluctuations on this scale are indeed observed on the CMB.