

Some common distributions

AIMS, Bayesian Statistics 2009

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1 Continuous distributions

1.1 Uniform

Notation $x \sim U(a, b)$

Parameters: Boundaries a, b , with $b > a$ ($-\infty < a < \infty$, $-\infty < b < \infty$)

Density: $f(x) = \frac{1}{b-a}$, for $x \in [a, b]$

Summary: $E[x] = \frac{a+b}{2}$, $Var(x) = \frac{(b-a)^2}{12}$, no mode

1.2 Normal

Notation: $x \sim N(\mu, \sigma^2)$

Parameters: Location μ ($-\infty < \mu < \infty$), scale $\sigma > 0$

Density $p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$ for $-\infty < x < \infty$

Summary: $E[x] = \mu$, $Var(x) = \sigma^2$, $mode(x) = \mu$

1.3 Chi-square (χ^2)

If $X \sim N(0, 1)$, then $X^2 \sim \chi_1^2$. Also, if X_1, X_2, \dots, X_p are all iid $\sim N(0, 1)$ then $\sum_{j=1}^p X_j^2 \sim \chi_p^2$.

Notation: $x \sim \chi_\nu^2$

Parameters: Degrees of freedom $\nu > 0$

Density: $f(x) = \frac{2^{\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}$ for $0 < x < \infty$

Summary: $E[x] = \nu$, $Var[x] = 2\nu$, $mode(x) = \nu - 2$ (for $\nu > 2$)

(The χ^2 is a special form of the Gamma distribution - what parameters of the Gamma distribution give you a χ^2 ?)

1.4 F distribution

Notation $x \sim F_{\nu_1, \nu_2}$

Parameters: Degrees of freedom ν_1 and ν_2 ($\nu_1 > 0, \nu_2 > 0$)

Density: $f(x) = \frac{\nu_1^{\nu_1/2} \nu_2^{\nu_2/2}}{B(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \frac{x^{\nu_1/2-1}}{(\nu_2 + \nu_1 x)^{(\nu_1 + \nu_2)/2}}$ for $0 < x < \infty$

Summary: $E[x] = \nu_2/(\nu_2 - 2)$, $Var(x) = \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}$, $mode(x) = \frac{\nu_2}{\nu_2 + 1} \frac{\nu_1 - 2}{\nu_1}$

Note if $W_1 \sim \chi_{\nu_1}^2$ and $W_2 \sim \chi_{\nu_2}^2$ then $X = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1, \nu_2}$

1.5 Exponential

Notation: $x \sim Expon(\theta)$

Parameters: Inverse scale θ where $\theta > 0$

Density: $f(x) = \theta e^{-\theta x}$ for $0 < x < \infty$

Summary: $E[x] = \frac{1}{\theta}$, $Var(x) = \frac{1}{\theta^2}$, $mode(x) = 0$

Notes: This is equivalent to a Gamma with $a = ?$ and $r = \theta$ Notes: The density can also be written in the form $\frac{1}{\lambda} e^{-\frac{1}{\lambda}x}$.

1.6 Gamma

Notation $x \sim Gamma(a, r)$

Parameters: Shape $a > 0$, rate $r > 0$ (the *rate* = $1/scale$ and might be called inverse scale)

Density: $f(x) = \frac{r^a}{\Gamma(a)} x^{(a-1)} e^{-xr}$ for $x > 0$.

Summary $E[x] = \frac{a}{r}$, $Var(x) = \frac{a}{r^2}$, $mode = \frac{a-1}{r}$ for $a \geq 1$

You can also see the Gamma parameterised slightly differently (with a scale rather than a rate)

Parameters: Shape $a > 0$, scale $s > 0$

Density $f(x) = \frac{(\frac{1}{s})^a}{\Gamma(a)} x^{(a-1)} e^{-\frac{x}{s}}$ for $x > 0$

1.7 Beta

Notation: $x \sim \text{Beta}(a, b)$

Parameters: “Prior sample size” $a > 0, b > 0$

Density: $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ for $0 \leq x \leq 1$

Summary: $E[x] = \frac{a}{a+b}, \text{Var}(x) = \frac{ab}{(a+b)^2(a+b+1)}, \text{mode}(x) = \frac{a-1}{a+b-2}$

1.8 Student t

Notation: $x \sim t_\nu(\mu, \sigma^2)$. t_ν is an abbreviation for $t_\nu(0, 1)$

Parameters: Degrees of freedom $\nu > 0$, location μ ($-\infty < \mu < \infty$), scale $\sigma > 0$.

Density: $f(x) = \frac{\Gamma((\nu+1)/2)}{\text{Gamma}(\nu/2)\sqrt{\nu\pi\sigma}} \left(1 + \frac{1}{\nu} \left(\frac{x-\mu}{\sigma}\right)^2\right)^{-(\nu+1)/2}$ for $-\infty < x < \infty$

Summary: $E[x] = \mu$ for $\nu > 1, \text{Var}(x) = \frac{\nu}{\nu-2}\sigma^2$ for $\nu > 2, \text{mode}(x) = \mu$

2 Discrete distributions

2.1 Binomial

Notation: $x \sim \text{Bin}(n, \pi)$

Parameters: π , probability of “success” ($\pi \in [0, 1]$). We also need the integer n .

Density: $f(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x}$; for $x = 0, 1, \dots, n$

Summary: $E[x] = n\pi, \text{Var}(x) = n\pi(1-\pi), \text{mode}(x) = \text{floor}\{(n+1)p\}$

2.2 Poisson

Notation: $x \sim \text{Poisson}(\lambda)$

Parameters: rate= $\lambda > 0$

Density: $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0, 1, 2, \dots$

Summary: $E[x] = \lambda, \text{Var}(x) = \lambda, \text{mode} = \text{floor}(\lambda)$

2.3 Negative Binomial

Notation: $x \sim NB(a, b)$

Parameters: Shape $a > 0$, inverse scale $b > 0$

Density: $f(x) = \binom{x+a-1}{a-1} \left(\frac{a}{a+1}\right)^a \left(\frac{1}{b+1}\right)^x$ for $x = 0, 1, 2, \dots$

Summary: $E[x] = \frac{a}{b}$, $Var(x) = \frac{a}{b}(b+1)$

2.4 Beta-binomial

Notation: $x \sim Beta - Bin(n, a, b)$

Parameters: "Prior sample size" $a > 0$, $b > 0$. We also need to know the integer n

Density: $f(x) = \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} \frac{\Gamma(a+x)\Gamma(n+b-x)}{\Gamma(a+b+n)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a)\Gamma(b)}$ for $x = 0, 1, 2, \dots, n$

Summary: $E[x] = n \frac{a}{a+b}$, $Var(x) = n \frac{ab(a+b+n)}{(a+b)^2(a+b+1)}$