School of Mathematics and Statistics

Conditional probability

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Overview: This webfile is designed as a revision aid to some introductory concepts in probability. It is intended to supplement a formal encouter with a text book or a set of lectures. These notes are meant to be slightly interactive, mysterious green dots, squares and boxes appear which you can click on to answer questions and check solutions.

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Aims

- To examine conditional probability
- To define the term “independence”
- To state Bayes Theorem

We start by quoting a table from Grimmett and Stirzaker (2004) which gives a full listing of set theory and probability theory notation.

Notation	Set theory	Probability Theory
Ω	A collection of objects	Sample space
ω	A subset of Ω	Elementary event/outcome
A	A subset of Ω	Event that some condition A occurs
A^C	Complement of A	No event A occurs
$A \cap B$	Intersection of A and B	Both A and B occur
$A \cup B$	Union of A and B	Either A occurs, or B occurs or both
$A \setminus B$	Difference	A occurs but B does not occur
$A \triangle B$	Symmetric difference	Either A or B but not both
$A \subseteq B$	Inclusion	If A occurs then B occurs
$\emptyset$	Empty set	Impossible event
Ω	Whole space	Certain event

1. Conditional probability

To date, we have only considered information obtained from:

(a) Specifying the sample space and

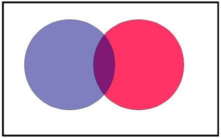
(b) explicitly or implicitly specifying a probability measure on that sample space

However, knowledge that a particular event has taken place can alter our assessment of the probability of other events. Consider a student who wishes to pass his course. However, this student doesn't like attending lectures. he is told that his chances of passing the course will increase if he attends lectures.

We can write this *conditional probability* as:

$$p[\text{Student will pass course}|\text{Student attends lectures}]$$

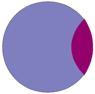
You have previously examined the situation where we could examine A (the event that the student attends lectures, the circle on the left), B (the event that the student passes, the circle on the right) as well as the intersection. We could draw a Venn diagram something like the one below:



However, once we know that the student has indeed attended lectures, the information we have on the problem changes the question. We are no longer interested in A^C, as we know the student attended. Hence, we *condition* our interest on A

Please ignore the fact that this diagram seems to show rather a lot of probability associated with the event that the student passes when he doesn't attend lectures - this is only a simple illustration;-)

A reduced diagram would look like the following. We therefore want to know the probability of the student passing, given that we know he attended lectures.



We can use the following statement:

Definition 1

$$p[A|B] = \frac{p[A \cap B]}{p[B]} \text{ if } p[B] > 0$$

to solve this problem. This expression is undefined if $p[B] = 0$

In frequency terms, this expression can be thought of as:

$$\frac{\text{Number of experiments in which both A and B occurred}}{\text{Number of experiments in which B occurred}}$$

Strictly speaking, all probability is conditional. Earlier in the course we could have referred to $p(A)$ as $p(A|\Omega)$.

Begin Quiz A manufacturer of aeroplane equipment knows from past experience that

(i) The probability that an order will be ready for shipment is $p[R] = 0.8$

(ii) The probability that an order will be ready for shipment *and* delivered on time is $p[D] = 0.72$

1. What is the probability that an order will be delivered on time *given* that it was ready on time?

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; shift + click to get the solution.

1.1. Axioms of conditional probability

As with probability, we only need a few axioms for conditional probability.

Definition 2 *Axioms of conditional probability:*

- $p[A|B] \geq 0$ *for any event* A
- $p[A|B] \leq 1$ *for any event* $A \supset B$
- *If* $A_1, A_2, \ldots$ *is a sequence of mutually exclusive events then:* $p[A_1 \cup A_2 \cup \ldots |B] = p[A_1|B] + p[A_2|B] + \ldots$
- *If* $B \supset A_1$ *and* $B \supset A_2$ *and* $p[B_2] \neq 0$ *then* $\frac{p[A_1|B]}{p[A_2|B]} = \frac{p[A_1]}{p[A_2]}$

It can be seen that conditional axioms (1), (2) and (3) are analogous to the axioms of probability. The fourth axiom tells us that the relative frequency A_1 versus A_2 remains the same after B has happened.

1.2. Consequences

A few small points worth mentioning:

- $p[B|B] = 1$
- $p[A^C|B] = 1 - p[A|B]$
- $p[A_1 \cup A_2|B] = p[A_1|B] + p[A_2|B] - p[A_1 \cap A_2|B]$

But we have an additional theorem that is specific to conditional probability and quite useful:

Definition 3 *For any pair of events A and B, where $p[B] > 0$:*

$$p[A|B] = \frac{p[A \cap B]}{p[B]}$$

1.3. Law of alternatives

Now we only know that one of several possible events A_1,A_2, ... has occured (there may be infinitely many of these alternatives).

If

- A_i $i \geq 1$ are pairwise disjoint
- $p(\bigcup_i A_i) = 1$ (we sometimes require the condition that $\bigcup_i A_i = \Omega$
- $p(A_i) > 0$ for every i

then

$$p(B) = p(B|A_1)P(A_1) + p(B|A_2)p(A_2) + \cdots$$

2. Bayes Theorem

This is rather an important theorem, and follows from the above. As stated here, this theorem is a well accepted argument.

Definition 4 *Let events $A_1, A_2, \ldots, A_n$ be mutually exclusive, such that $p[A_1 \cup A_2 \cup \ldots \cup A_n] = 1$ and $p[A_i] > 0$ for each i. Let B be an event such that $p[B] > 0$. Then:*

$$p[A_i|B] = \frac{p[B|A_i]p[A_i]}{\sum_{j=1}^{n} p[B|A_j]p[A_j]}, i = 1, 2, \ldots, n$$

As stated, Bayes theorem is a very simple and widely accepted result in conditional probability. Occasionally we use the following terminology:

- Prior probability: $p[A_i]$
- Posterior probability $p[A_i|B]$

Where Bayes theorem starts to get interesting is when we no longer use simple probabilities in the calculations. Specifically, when we start using the likelihood as $p[B|A_i]$ we can carry out some very interesting inference.

Begin Quiz Consider the following table, concerning the proportion of output produced by one of four machines, as well as the proportion of total defectives generated by these machines.

Machine name	Production (%)	Defectives (%)
Paul	15	4
Ringo	30	3
John	20	5
George	35	2

1. If a screw is picked at random from the inventory, what is the probability that it will be defective (please answer to 3 decimal places):

2. If a randomly picked screw was defective, what is the probability that it was produced by John (i.e. find $p[John|D]$). (3 d.p.)

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; Shift ⇑ + click to get the solution.

Here are a number of related applications, all using the same formula ($p[A|B] = \frac{p[A \cap B]}{p[B]}$)

Begin Quiz

1. The probability that a student reaches level 3 on a computer game if it is noisy in the house is 0.4. The corresponding probability, that a student reaches level 3 on a computer game if the house is quiet is 0.7. The probability that the house is quiet on a given day is 0.3. On a day chosen at random, what is the probability that the house is quiet, given the student has reached level 3.(please answer to 2 decimal places):

2. Alcohol tests are regularly conducted among drivers in a particular region. Drivers are first subjected to a breath test. Only after a positive breath test result is a driver taken for a blood test. The breath test yields a positive result among 90% of drunken drivers, and a positive result among 5% of sober drivers. If we believe that one out of every hundred drivers on the roads in the region is driving under the influence of alcohol, find the probability that a randomly selected driver who shows a positive breath test is sober.

3. A shop owner is willing to cash personal cheques for amounts up to Euro 50, but has become wary of customers who wear sunglasses. 50% of cheques written by sunglasses wearers are returned by the bank unpaid ("bounced"). In contrast, 98% of cheques written by non-sunglasses wearers are accepted by the bank. The shopkeeper estimates that 10% of customers wear sunglasses. If a bank returns a cheque unpaid, what is the probability that it was written by a sunglasses wearing customer? Please answer to 4 decimal places.

End Quiz | Correct | Points: | Score:

Click on the Ans box to get the correct answer; Shift ⇑ + click to get the solution.

3. Independence

What about the situation whereby the probability $p[A]$ is unchanged by our knowledge that B has occurred, i.e., that $p[A] = p[A|B]$. In this case, we regard A and B as being *independent*. This is only defined when $p[B] > 0$.

Definition 5 *Events A and B are called* independent *if:*

$$p[A] = p[A|B]$$

Noting that the result $P[A|B] = p[A \cap B|B] + p[A \cap B^C|B]$ simplifies (the last term is $\emptyset$ gives us:$p[A|B] = p[A \cap B|B]$ which leads us to another definition of independence.

Definition 6 *Two events can be called independent if:*

$$p[A \cap B] = p[A] \times p[B]$$

and more formally, a family of events $\{A_i : i \in I\}$ *is called* independent *if:*

$$p\left[\bigcap_{i \in J} A_i\right] = \prod_{i \in J} p[A_i]$$

for all finite subsets J *of* I

- Remember that if events are mutually exclusive then $p[A|B] = 0$.
- Do note that it is *false* to claim that if $A \cap B = \emptyset$ then events are independent.

Quiz Roads There are two roads connecting town A and town B, and a further two roads connecting town B and town C. Each of the four roads has a probability p of being blocked by snow. What is the probability that there is an open road from A to C

(a) p^4

(b) $(1 - p^2)^2$

(c) $1 - p^4$

(d) $(1 - p)^4$

Solutions to Quizzes

Solution to Quiz: We need to find:

$$p[D|R] = \frac{p[D \cap R]}{p[R]} = \frac{0.72}{0.80} = 0.9 \qquad (1)$$

.

Click on that green button to return to the quiz →

Solution to Quiz: We solve this problem using:

$$\begin{aligned} p[D] &= p[D|Paul]p[Paul] + p[D|Ringo]p[Ringo] \\ &\quad + p[D|John]p[John] + p[D|George]p[George] \\ &= 0.04 \cdot 0.15 + 0.03 \cdot 30 + 0.05 \cdot 0.20 + 0.02 \cdot 0.35 \\ &= 0.032 \end{aligned}$$

■

Solution to Quiz: We need to use the result: $p[John|D] = \frac{p[John \cap D]}{p[D]}$.
We just worked out $p[D] = 0.032$, in doing that we should have noticed that
$p[John \cap D] = 0.01$. Hence we need
$p[John|D] = \frac{0.01}{0.032} = 0.31$ ■

Solution to Quiz: As above, first we need to find the denominator:
$p[Level3] = p[Level3|Q] + p[Level3|Noisy] = 0.4 \times 0.7 + 0.7 \times 0.3 = 0.49$.
We then need to use:
$p[Q|Level3] = \frac{p[Level3|Q]p[Quiet]}{p[Level3]}$
which uses $\frac{0.21}{0.49} = 0.43$

Thanks to Prince for pointing out an error in this solution! ■

Solution to Quiz: This is interesting. It follows the previous examples exactly.

We first need to know $p[Fail] = p[Fail|Sober] + p[Fail|Drunk]$
$= 0.9 \times 0.01 + 0.05 \times 0.99 = 0.0585$.

This suggests that we can anticipate 0.06 tests (6%) to yield a positive result (i.e., one needing confirmation by a blood test). However, when we complete the calculation, we find that

$p[Sober|Fail] = \frac{p[Sober \cap Fail]}{p[Fail]} = \frac{0.0495}{0.0585} = 0.8462$ ■

Solution to Quiz: We first need to know $p[Bounce] = p[Bounce|Sunglasses$
$p[Bounce|Nosunglasses]$
$= 0.5 \times 0.10 + 0.02 \times 0.90 = 0.0680.$

This suggests that we can anticipate 0.068 (7%) of cheques to be returned by the bank. However, when we complete the calculation, we find that
$p[Sunglasses|Bounce] = \frac{p[Sunglasses \cap Bounce]}{p[Bounce]} = \frac{0.0180}{0.0680} = 0.2647$

Solution to Quiz: We wish to define a probability for:

$$\begin{aligned} p[Open] &= p[\text{A and B open}] \cap p[\text{B and C open}] \\ &= p[\text{A and B open}] \times p[\text{B and C open}] \\ &= (1 - p[\text{A and B closed}] \times p[\text{B and C closed}]) \end{aligned}$$

Note the (rather unlikely) statement that p is the same for all roads, hence this simplifies:

$$\begin{aligned} p[Open] &= (1 - p[\text{A and B closed}])^2 \\ &= (1 - p[\text{A and B (1) closed} \cap \text{A and B (2) closed}])^2 \\ &= (1 - p[\text{A and B (1) closed}] \times p[\text{A and B (2) closed}])^2 \\ &= (1 - p^2)^2 \end{aligned}$$

■