Homework 2

Due date: March 10th, 2010

1. Consider the abelian Higgs model in 1+1 dimensions as was treated in the lectures. The Euclidean action is:

$$S_E = \int d\tau dx \; ((\partial_\mu + iA_\mu)\phi)^*(\partial_\mu + iA_\mu)\phi + \frac{\lambda}{4}(|\phi|^2 - a^2)^2 + \frac{1}{4e^2}F_{\mu\nu}F_{\mu\nu} \tag{1}$$

   (a) For the vortex ansatz written in polar coordinates (r, θ):

$$\phi = e^{i\theta} f(r) \tag{2}$$

$$A_\mu = \frac{\epsilon_{\mu\nu} x_\nu \Phi(r)}{r^2} \tag{3}$$

   find the equations of motion for $f(r)$ and $\Phi(r)$.

   (b) Using the dilute instanton gas approximation, as done in the lectures, calculate the expectation value

$$\langle\zeta|\frac{1}{2}\epsilon_{\sigma\rho}F_{\sigma\rho}|\zeta\rangle = \frac{\int_{all\ \nu} \mathcal{D}(\phi, A_\mu) e^{-S_E/\hbar} e^{i\nu\zeta} \frac{1}{2}\epsilon_{\sigma\rho}F_{\sigma\rho}}{\int_{all\ \nu} \mathcal{D}(\phi, A_\mu) e^{-S_E/\hbar} e^{i\nu\zeta}}. \tag{4}$$

   Use the fact that the expectation value should be independent of position, hence $\langle\zeta|\frac{1}{2}\epsilon_{\sigma\rho}F_{\sigma\rho}|\zeta\rangle = (1/L\beta)\langle\zeta|\int d\tau dx\ \frac{1}{2}\epsilon_{\sigma\rho}F_{\sigma\rho}|\zeta\rangle$.

2. We want to define the following integral by analytic continuation and determine its imaginary part in first approximation as $\kappa \to \infty$

$$\mathcal{I}(\kappa, \alpha) = \int_0^\infty dz\ e^{\kappa(z-3\alpha^2)^3(z+2\alpha^2)^2} \tag{5}$$

   (a) Find the critical points of the exponent. Are there any singularities for finite values of z?

   (b) Define the analytic continuation by deforming the contour of integration into the first quadrant. For the integral to converge for real, positive κ which direction in the complex plane must the contour approach asymptotically?

   (c) Find the path of steepest descent from the origin, along the real z axis to the next critical point, and then following the line of steepest descent into the upper half plane. What is the direction in which the path of steepest descent leaves the real z axis at the second critical point?

   (d) Compute the imaginary part of the integral in the Gaussian approximation.