Homework

Due date: March 3rd, 2010

1. We want to find the path integral representation for the Euclidean persistence amplitude for a particle at the origin to remain at the origin after a Euclidean time β for the time dependent Hamiltonian:

$$\hat{h}(\hat{X},\hat{P},\tau)=\frac{1}{2}\left(\hat{P}^2+V''(\bar{z}(\tau))\hat{X}^2\right) \tag{1}$$

A state $|z=0\rangle$ corresponding to a particle at position $z=0$ at $\tau=0$ will "evolve" in Euclidean time β to the state

$$\mathcal{U}(\beta,0)|z=0\rangle = T\left(e^{-\int_0^\beta d\tau \frac{1}{\hbar}\hat{h}(\hat{X},\hat{P},\tau)}\right)|z=0\rangle \tag{2}$$

where $\mathcal{U}(\tau,0)$ satisfies the equation

$$\frac{d}{d\tau}\mathcal{U}(\tau,0)=-\frac{1}{\hbar}\hat{h}(\hat{X},\hat{P},\tau)\mathcal{U}(\tau,0). \tag{3}$$

   (a) Show the linear approximation for $\mathcal{U}(\tau+\epsilon,\tau)$ is given by

$$\mathcal{U}(\tau+\epsilon,\tau)\approx 1-\frac{\epsilon}{\hbar}\hat{h}(\hat{X},\hat{P},\tau) \tag{4}$$

   (b) Split the Euclidean time interval into $N+1$ equal parts and use the linear approximation and the method used in the lectures (inserting complete sets of position and momentum eigenstates in between the linear approximations) to show

$$\langle z=0|T\left(e^{-\int_0^\beta d\tau \frac{1}{\hbar}\hat{h}(\hat{X},\hat{P},\tau)}\right)|z=0\rangle=\mathcal{N}\int\mathcal{D}z(t)e^{-\frac{1}{\hbar}\int_0^\beta dt\left(\frac{1}{2}(\dot{z}(\tau))^2+V''(\bar{z}(\tau)z(\tau)^2)\right)} \tag{5}$$

   where the path integral is over all paths that satisfy $z(0)=0=x(\beta)$.

2. Consider the action, in real time, and the corresponding Lagrangian

$$S_M=\int dt\mathcal{L}_M=\int dt\left(\sum_{i=1}^N\left(|(\partial_\tau+iA(t))\phi_i(t)|^2-m^2|\phi_i(t)|^2\right)-\lambda A(t)\right) \tag{6}$$

where λ and m are constants, A is a real valued function of t and $\phi_i(t)$ are N, complex valued functions of time.

(a) Show that under analytic continuation to Euclidean time $t \to -i\tau$, with the added continutation $A(t) \to iA(\tau),\ \phi_i(t) \to \phi_i(\tau)$, the Euclidean action is given by

$$S_E = \int d\tau \mathcal{L}_E = \int d\tau \left(\sum_{i=1}^{N} \left(|(\partial_\tau + iA(\tau))\phi_i(\tau)|^2 + m^2 |\phi_i(\tau)|^2 \right) + i\lambda A(\tau) \right). \quad (7)$$

This is a surprising result, the Euclidean action is not a real positive functional. It has a non-zero complex part.

(b) Rewrite the Euclidean Lagrangian completely in terms of the real variables, x_i, y_i and A, where $\phi_i = x_i + iy_i$ and find the Euclidean equations of motion.

(c) Show that the non-trivial solutions of the equations of motion (ie. solutions for which not all fields vanish) occur only for complex values of the fields x_i, y_i and A.