Solution - Homework 2

(1): $S_E = \int d^2x \, |(\partial_\mu + iA_\mu)\varphi|^2 + \frac{\lambda}{4}(|\varphi|^2 - a^2)^2 + \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu}$

$$\varphi = e^{i\theta} f(r) \qquad A_\mu = \frac{\epsilon_{\mu\nu} x_\nu}{r^2} \Phi(r)$$

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 \qquad\qquad F_{\mu\nu}F_{\mu\nu} = 2(F_{12})^2$$

$$= \partial_1\left(\epsilon_{21}\frac{x_1}{r^2}\Phi(r)\right) - \partial_2\left(\epsilon_{12}\frac{x_2}{r^2}\Phi(r)\right)$$

$$= -\left(\partial_1\left(\frac{x_1}{r^2}\right)\Phi(r) + \frac{x_1}{r^2}\partial_1\Phi + \partial_2\left(\frac{x_2}{r^2}\right)\Phi(r) + \frac{x_2}{r^2}\partial_2\Phi\right)$$

$$= -\left(\left(\underbrace{\frac{1}{r^2} - 2\frac{x_1^2}{r^3} + \frac{1}{r^2} - 2\frac{x_2^2}{r^3}}_{=0}\right)\Phi + \left(\frac{x_1^2 + x_2^2}{r^3}\right)\partial_r\Phi(r)\right)$$

$$= -\frac{1}{r}\partial_r\Phi(r)$$

$$\frac{\lambda}{4}(|\varphi|^2 - a^2)^2 = \frac{\lambda}{4}(f(r)^2 - a^2)^2$$

$$(\partial_1 + iA_1)\varphi = \left(\partial_1 + i\epsilon_{12}\frac{x_2}{r^2}\Phi(r)\right)e^{i\theta} f(r)$$

$$= e^{i\theta}\left(i\partial_1\theta + i\frac{x_2}{r^2}\Phi(r) + \partial_1\right) f(r) \qquad\qquad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\cancel{= e^{i\theta}\, i \frac{1}{\frac{x_2^2}{x_1^2}+1}\left(-\frac{x_2}{x_1^2}\right)}$$

$$= e^{i\theta}\left(i\frac{1}{\left(\frac{x_2}{x_1}\right)^2 + 1}\left(-\frac{x_2}{x_1^2}\right) + i\frac{x_2}{r^2}\Phi(r) + \partial_1\right) f(r)$$

$$= e^{i\theta}\left(i\frac{x_2}{r^2}(\Phi(r) - 1) f(r) + \partial_1 f(r)\right)$$

so $\quad |(\partial_1 + iA_1)\varphi|^2 = \frac{x_2^2}{r^4}(\Phi(r)-1)^2 f^2(r) + (\partial_1 f(r))^2$

clearly $\quad |(\partial_2 + iA_2)\varphi|^2 = \frac{x_1^2}{r^4}(\Phi(r)-1)^2 f^2 + (\partial_2 f(r))^2$

so $\quad |D_\mu \varphi|^2 = \frac{(\Phi(r)-1)^2}{r^2} f^2(r) + \cancel{\partial_\mu} (\partial_1 f(r))^2 + (\partial_2 f(r))^2$

$$= \frac{(\Phi(r)-1)^2}{r^2} f^2(r) + (\partial_r f)^2$$

So

$$S = 2\pi \int_0^\infty dr\, r \left(\frac{(\Phi(r)-1)^2}{r^2} f^2(r) + (\partial_r f)^2 + \frac{1}{2e^2} \frac{1}{r^2} (\partial_r \Phi(r))^2 \right.$$

$$\left. + \frac{\lambda}{4}(f^2 - a^2)^2 \right)$$

$\frac{\partial S}{\partial f'} = r 2 f' \qquad \frac{\partial S}{\partial f} = 2\frac{(\Phi(r)-1)^2}{r} f + r\lambda (f^2 - a^2) f$

giving $\quad \partial_r (r 2 f') - 2 \frac{(\Phi(r)-1)^2}{r} f - r\lambda (f^2 - a^2) f = 0$

ie

$$-\frac{1}{r}\partial_r (r\, \partial_r f(r)) + \left(\frac{\Phi(r)-1}{r}\right)^2 f(r) + \frac{\lambda}{2}(f^2 - a^2) f = 0$$

which up to redefinition of λ, a & $\Phi \to \frac{\Phi}{2\pi}$ the same as in the notes and

$$\frac{dS}{d\Phi'(r)} = \frac{2}{e^2}\frac{\Phi'(r)}{r} \qquad \frac{dS}{d\Phi} = \frac{2(\Phi(r)-1)}{r} f^2$$

so eq. of motion is

$$-\frac{1}{e^2}\partial_r \frac{1}{r}\partial_r \Phi + \frac{(\Phi(r)-1)}{r} f^2 = 0$$

b) Find $\langle \zeta | \frac{1}{2}\epsilon_{\sigma\rho} F_{\sigma\rho} | \zeta \rangle$

$$= \frac{1}{L\beta} \langle \zeta | \int d^2z \frac{1}{2}\epsilon_{\sigma\rho} F_{\sigma\rho} | \zeta \rangle$$

$$= \frac{1}{L\beta} \frac{\int \mathcal{D}(\varphi, A)\, e^{i\nu\zeta}\, e^{-S_E/\hbar}\, 2\pi\nu}{\int \mathcal{D}(\varphi, A)\, e^{i\nu\zeta}\, e^{-S_E/\hbar}}$$

$$= \frac{1}{L\beta} \frac{2\pi}{i} \frac{d}{d\zeta} \ln\left(\int \mathcal{D}(\varphi, A)\, e^{i\nu\zeta}\, e^{-S_E/\hbar}\right)$$

$$= \frac{1}{L\beta} \frac{2\pi}{i} \frac{d}{d\zeta} \ln e^{-E(\zeta)\beta/\hbar}$$

$$= \frac{1}{L\beta} \frac{2\pi}{i} \frac{d}{d\zeta} \left(2\beta L\, e^{-\beta S_0/\hbar} \cos\zeta \right)$$

$$\boxed{\langle \zeta | \tfrac{1}{2}\epsilon F | \zeta \rangle = 4\pi i\, e^{-\beta S_0/\hbar} \sin\zeta}$$

② $J(X,\alpha) = \int_0^\infty dz \; e^{X(z-3\alpha^2)^3(z+2\alpha^2)^2} = \int_0^\infty dz \, e^{X f(z)}$

$$f(z) = (z-3\alpha^2)^3(z+2\alpha^2)^2$$

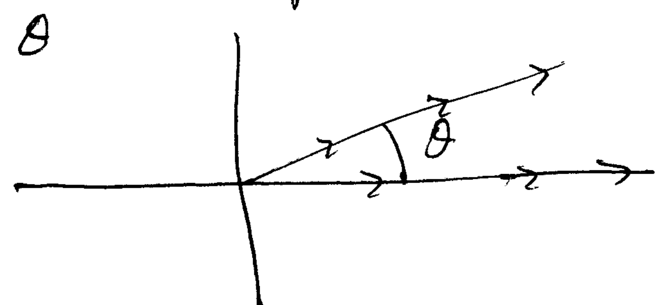
$$f'(z) = 3(z-3\alpha^2)^2(z+2\alpha^2)^2 + (z-3\alpha^2)^3 2(z+2\alpha^2)$$

$$= (z-3\alpha^2)^2(z+2\alpha^2)\left(3(z+2\alpha^2) + 2(z-3\alpha^2)\right)$$

$$= (z-3\alpha^2)^2(z+2\alpha^2)\, 5z$$

(a) $f'(z_0) = 0 \Rightarrow z_0 = 3\alpha^2, \ 0, \ -2\alpha^2$

(b) The original integral is defined for $\mathrm{Re}(X)$ negative. We deform the contour into the first quadrant along a ray of angle θ



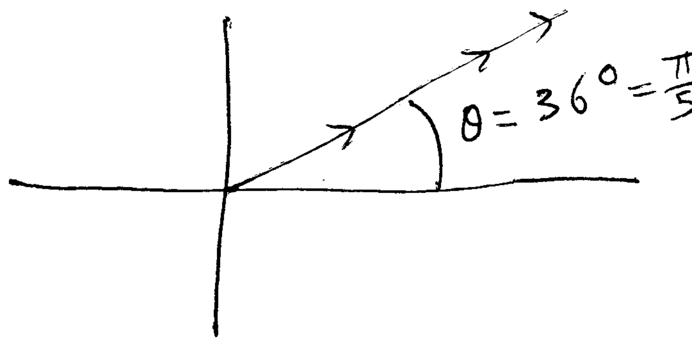
So $\arg(X) < (\frac{\pi}{2}, \frac{3\pi}{2})$ is shifted by 5θ since asymptotically $f(z) \sim z^5$.

So we need $\arg(X e^{i5\theta}) = \arg(X + 5\theta) \in (\frac{\pi}{2}, \frac{3\pi}{2})$

ie $\arg(x) \in \left(\frac{\pi}{2} - 5\theta,\ \frac{3\pi}{2} - 5\theta\right)$

thus for $\theta = \frac{\pi}{5}$ we get $\arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

ie $\underline{Re(x) > 0}$.



(c) Path of steepest descent follows the real z axis from $z = z_0 = 0$ to $z = z_0 = 3x^2$ and then turns into the upper complex plane. The critical point at $z = z_0 = 3x^2$ is not regular

$$f''(3x^2) = 2(z-3x^2)(z+2x^2)5z + (z-3x^2)^2\frac{d}{dz}(z+2x^2 5z)\Big|_{z=} = 0$$

but

$$f'''(3x^2) = 2(z+2x^2)5z\Big|_{z=3x^2} = 2\cdot 5x^2\cdot 15x^2 = 150x^4$$

So

$$f(z) = \overset{=0}{f(3x^2)} + \underset{=0}{f'(3x^2)}(z-3x^2) + \frac{1}{2}\underset{=0}{f''(3x^2)}(z-3x^2)^2 + \frac{1}{\ } f'''(3x^2)(z-3x^2)^3 + \cdots$$

so $$f(z) = \frac{1}{6}\, 150\lambda^4 \left(z - 3\lambda^2\right)^3 + \cdots$$

Let $$x + iy = z - 3\lambda^2$$

so $$(z-3\lambda^2)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$
$$= x^3 - 3xy^2 + i\left(3x^2y - y^3\right)$$

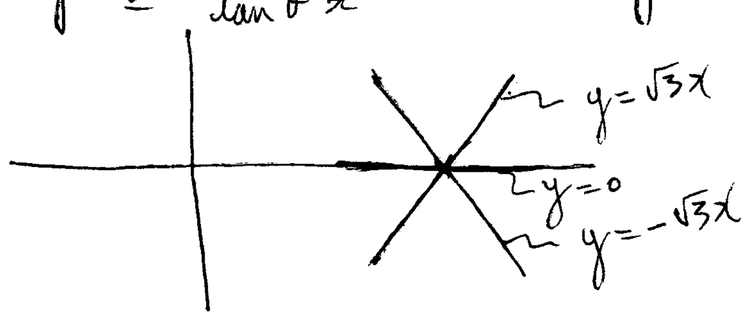
$$\operatorname{Im}\left((z-3\lambda^2)^3\right) = 3x^2y - y^3 = 0$$ for the path of steepest descent (const. imaginary part)
(actually ascent!)

so $$(3x^2 - y^2)\, y = 0$$

ie $y = 0$ or $3x^2 - y^2 = 0$

ie $$(\sqrt{3}\,x - y)(\sqrt{3}\,x + y) = 0$$

$y = \sqrt{3}\,x$ or $y = -\sqrt{3}\,x$
$= \tan\theta\, x$



$\tan\theta = \sqrt{3}$
ie $\sin\theta = \frac{\sqrt{3}}{2}$
$\cos\theta = \frac{1}{2}$
$\theta = 60^\circ = \pi/3$

So the integral is

$$I(X,\alpha) \simeq \int_0^{3\alpha^2} dx\, e^{X(x-3\alpha^2)^3(x+2\alpha^2)^2} + \int_C dz\, e^{X(z-3\alpha^2)^3(z+2\alpha}$$

where C is the contour starting at $z = 3\alpha^2$ and heading into the complex plane with angle $60^\circ = \pi/3$

For large X the dominant contribution comes from near $z=3$

On C $\quad z = 3\alpha^2 + x + iy$

with $x = s \quad y = \sqrt{3}\, s$

s the parameter along the curve

$$f(3\alpha^2+x+iy) \simeq \frac{1}{6} f'''(3\alpha^2)(x+iy)^3 \Big|_{\substack{x=s \\ y=\sqrt{3}s}} = \frac{1}{6}\, 150\alpha^4 (1+i\sqrt{3})^3 s$$

$$= 25\alpha^4\left(1+3i\sqrt{3}+3(-3) - i\,3\sqrt{3}\right)s^3$$

$$= 25\alpha^4(-8)s^3 = -200\alpha^4 s^3$$

and

$$dz = (1+i\sqrt{3})\,ds$$

so

$$I(X,\alpha) \simeq \int_0^{3\alpha^2} dx\, e^{X(x-3\alpha^2)^3(x+2\alpha^2)^2} + (1+i\sqrt{3})\int_0^\infty ds\, e^{-200\alpha^4 s^3}$$

$$\text{Imag}\, I(X,\alpha) = \sqrt{3}\int_0^\infty ds\, e^{-200\alpha^4 s^3} = \sqrt{3}\,\big(3\,\Gamma(\tfrac{4}{3})$$