Homework 1 Solutions

$$\hat{h} = \frac{1}{2}\left(\hat{p}^2 + V''(\bar{z}(\tau))\,\hat{X}^2\right)$$

$$\mathcal{U}(\beta, 0) = T\, e^{-\frac{1}{\hbar}\int_0^\beta d\tau'\, \hat{h}(\hat{X}, \hat{P}, \tau')}$$

which is defined by

$$\frac{d}{d\tau}\,\mathcal{U}(\tau, 0) = -\frac{1}{\hbar}\,\hat{h}(\hat{X}, \hat{P}, \tau)\,\mathcal{U}(\tau, 0)$$

a) Show $\quad \mathcal{U}(\tau+\epsilon, \tau) \simeq 1 - \frac{\epsilon}{\hbar}\,\hat{h}(\hat{X}, \hat{P}, \tau)$

We must define $\mathcal{U}(\tau', \tau^0)$ where $\tau^0 \leq \tau'$.

The only reasonable definition is

$$\mathcal{U}(\tau', \tau^0)\,\mathcal{U}(\tau^0, 0) = \mathcal{U}(\tau', 0)$$

ie $\quad \mathcal{U}(\tau', \tau^0) = \mathcal{U}(\tau', 0)\left(\mathcal{U}(\tau^0, 0)\right)^{-1}$

Then $\quad \mathcal{U}(\tau, \tau) = \mathcal{U}(\tau, 0)\left(\mathcal{U}(\tau, 0)\right)^{-1} = 1$

So

$$\mathcal{U}(\tau+\epsilon, \tau) = \mathcal{U}(\tau, \tau) + \frac{d}{d\epsilon}\,\mathcal{U}(\tau+\epsilon, \tau)\Big|_{\epsilon=0}\,\epsilon + \cdots$$

$$= 1 + \epsilon\,\frac{d}{d\tau'}\,\mathcal{U}(\tau', \tau)\Big|_{\tau'=\tau}$$

But $\quad \cancel{\#} = 1 + \epsilon\,\frac{d}{d\tau'}\,\mathcal{U}(\tau', 0)\left(\mathcal{U}(\tau, 0)\right)^{-1}\Big|_{\tau'=\tau}$

$$= 1 + \epsilon\left(-\frac{1}{\hbar}\,\hat{h}(\hat{X},\hat{P},\tau')\,\mathcal{U}(\tau',0)\left(\mathcal{U}(\tau,0)\right)^{-1}\Big|_{\tau'=\tau}\right) +$$

$$= 1 + \epsilon\left(\frac{-1}{\hbar}\right)\hat{h}(\hat{X},\hat{P},\tau)\,\underbrace{\mathcal{U}(\tau,0)\left(\mathcal{U}(\tau,0)\right)^{-1}}_{=\mathbb{1}}$$

$$\simeq 1 - \frac{\epsilon}{\hbar}\,\hat{h}(\hat{X},\hat{P},\tau)$$

b)

$$\langle z=0|\,\mathcal{U}(\beta,0)\,|z=0\rangle = \langle z=0|\,\mathcal{U}(\beta,\beta_N)\,\mathcal{U}(\beta_N,\beta_{N-1})\cdots\mathcal{U}(\beta_2,\beta_1)$$
$$\times\ \mathcal{U}(\beta_1,0)\,|z=0\rangle$$

where $\beta_n = \frac{n\beta}{N+1}$, ~~defining~~ observing $\beta = \beta_{N+1}$, $\beta_0 = 0$

~~we have~~ and inserting a complete set of states in between each operator $\mathcal{U}(\beta_i,\beta_{i-1})$ we have

$$= \int dz_1\cdots dz_N\,\langle z=0|\,\mathcal{U}(\beta_{N+1},\beta_N)\,|z_N\rangle\langle z_N|\,\mathcal{U}(\beta_N,\beta_{N-1})\,|z_{N-1}\rangle$$
$$\cdots\ |z_2\rangle\langle z_2|\,\mathcal{U}(\beta_2,\beta_1)\,|z_1\rangle\langle z_1|\,\mathcal{U}(\beta_1,\beta_0)\,|z=0\rangle$$

$$= \int \prod_{n=0}^{N} \langle z_{n+1}|\,\mathcal{U}(\beta_{n+1},\beta_n)\,|z_n\rangle$$

Then

$$\langle z_{n+1}|\, \mathcal{U}(\beta_{n+1},\beta_n)\,|z_n\rangle$$

$$\simeq \langle z_{n+1}|\, 1 - \frac{\epsilon}{\hbar} \hat{h}(\hat{X},\hat{P},\beta_n)\,|z_n\rangle$$

$$= \langle z_{n+1}|\, 1 - \frac{\epsilon}{\hbar}\left(\frac{1}{2}\left(\hat{P}^2 + V''(\bar{z}(\beta_n))\,\hat{X}^2\right)\right)|z_n\rangle$$

$$= \langle z_{n+1}|\, 1 - \frac{\epsilon}{\hbar}\left(\frac{1}{2}\hat{P}^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\,z_n^2\right)|z_n\rangle$$

$$= \int dp_n\, \langle z_{n+1}|\, 1 - \frac{\epsilon}{\hbar}\left(\frac{1}{2}\hat{P}^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\,z_n^2\right)|p_n\rangle\langle p_n|z_n\rangle$$

$$= \int dp_n\, \langle z_{n+1}|\, 1 - \frac{\epsilon}{\hbar}\left(\frac{1}{2}p_n^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\,z_n^2\right)|p_n\rangle\langle p_n|z_n\rangle$$

$$\simeq \int dp_n\, e^{-\frac{\epsilon}{\hbar}\left(\frac{1}{2}p_n^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\,z_n^2\right)} \langle z_{n+1}|p_n\rangle\langle p_n|z_n\rangle$$

$$= \int dp_n\, e^{-\frac{\epsilon}{\hbar}\frac{1}{2}\left(p_n^2 + V''(\bar{z}(\beta_n))\,z_n^2\right)} \frac{e^{i\frac{p_n}{\hbar}(z_{n+1}-z_n)}}{2\pi\hbar}$$

So $\langle z=0|\, \mathcal{U}(\beta,0)\,|z=0\rangle = \int \prod_{n=1}^{N} \frac{dz_n\,dp_n}{2\pi\hbar}\, e^{-\frac{\epsilon}{\hbar}\frac{1}{2}\left(p_n^2 + V''(\bar{z}(\beta_n))\,z_n^2\right)} \times$

$$\times\, e^{i p_n \frac{(z_{n+1}-z_n)}{\hbar}}$$

$$= \int \prod_{n=1}^{N} \frac{dz_n\,dp_n}{2\pi\hbar}\, e^{-\frac{\epsilon}{\hbar}\left(\frac{1}{2}p_n^2 - i p_n\left(\frac{z_{n+1}-z_n}{\epsilon}\right) - \frac{1}{2}\left(\frac{z_{n+1}-z_n}{\epsilon}\right)^2\right)} \times$$

$$\times\, e^{-\frac{\epsilon}{\hbar}\left(\frac{1}{2}\left(\frac{z_{n+1}-z_n}{\epsilon}\right)^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\,z_n^2\right)}$$

$$= \int \prod_{n=1}^{N} \frac{dz_n dp_n}{2\pi\hbar} e^{-\frac{\epsilon}{\hbar}\frac{1}{2}\left(p_n - i\left(\frac{z_{n+1}-z_n}{\epsilon}\right)\right)^2} e^{-\frac{\epsilon}{\hbar}\left(\frac{1}{2}\left(\frac{z_{n+1}-z_n}{\epsilon}\right)^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\right.}$$

$$= \int \prod_{n=1}^{N} \frac{dz_n}{\cancel{\sqrt{2\pi\hbar}}} \sqrt{\frac{2\pi\hbar}{\epsilon}} \; e^{-\frac{\epsilon}{\hbar}\left(\frac{1}{2}\left(\frac{z_{n+1}-z_n}{\epsilon}\right)^2 + \frac{1}{2}V''(\bar{z}(\beta_n))\, z_n^2\right)}$$

$$\longrightarrow = \mathcal{N} \int_{z(\pm\beta/2)=0} \mathcal{D}z(\tau)\; e^{-\int_0^\beta d\tau \left(\frac{1}{2}(\dot{z}(\tau))^2 + \frac{1}{2}V''(\bar{z}(\tau))\, z_n^2\right)}$$

as required.

(2) $$S_M = \int dt \sum_i \left[|(\partial_t + iA(t))\phi_i(t)|^2 - m^2|\phi_i(t)|^2 - \lambda A(t) \right.$$
$$= \int dt \sum_i \left[(\partial_t - iA(t))\phi_i^*(t)(\partial_t + iA(t))\phi_i(t) - m^2\phi_i^*(t)\phi_i(t) - \lambda A(t) \right.$$

a) $t \to -i\tau$ + $A(t) \to iA\tau$

We get $dt \to -i\,d\tau$

$$S_M \to -i\int d\tau \sum_i \Big(i(\partial_\tau - iA(\tau))\phi_i^*(\tau)\; i(\partial_\tau + iA(\tau))\phi_i(\tau)$$
$$- m^2\phi_i^*(\tau)\phi_i(\tau) - \lambda i A(\tau) \Big)$$

$$= i\int d\tau \sum_i (\partial_\tau - iA(\tau))\phi_i^*(\tau)(\partial_\tau + iA(\tau))\phi_i(\tau) + m^2|\phi_i(\tau)|^2$$
$$+ i\lambda A(\tau)$$

$$= i S_E$$

So
$$S_E = \int d\tau \sum_i |(\partial_\tau + i A(\tau))\phi_i(\tau)|^2 + m^2 |\phi_i(\tau)|^2 + i\lambda A$$

$$\phi_i(\tau) = x_i(\tau) + i\, y_i(\tau)$$

$$(\partial_\tau + i A(\tau))(x_i(\tau) + i\, y_i(\tau)) = \partial_\tau x_i - A(\tau)\, y_i(\tau)$$
$$+ i(\partial_\tau y_i(\tau) + A\, x_i(\tau))$$

$$|(\partial_\tau + i A(\tau))\phi_i \tau|^2 = (\partial_\tau x_i - A(\tau) y_i)^2 + (\partial_\tau y_i + A(\tau) x_i(\tau$$
$$= (\partial_\tau x_i)^2 + (\partial_\tau y_i)^2 - 2A(y_i \partial_\tau x_i - x_i \partial_\tau y_i)$$
$$+ A^2(x_i^2 + y_i^2)$$

So

$$S_E[x_i, y_i, A] = \int d\tau \sum_i (\partial_\tau x_i)^2 + (\partial_\tau y_i)^2 + m^2(x_i^2 + y_i^2)$$
$$- 2A(y_i \partial_\tau x_i - x_i \partial_\tau y_i) + A^2(x_i^2 + y_i^2) + i\lambda A(\tau)$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\tau x_i} = 2\partial_\tau x_i - 2A y_i \qquad \frac{\partial \mathcal{L}}{\partial x_i} = 2m^2 x_i + 2A \partial_\tau y_i + 2A^2 x_i$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\tau y_i} = 2\partial_\tau y_i + 2A x_i \qquad \frac{\partial \mathcal{L}}{\partial y_i} = 2m^2 y_i - 2A\partial_\tau x_i + 2A^2 y_i$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\tau A} = 0 \qquad \frac{\partial \mathcal{L}}{\partial A} = -2\sum_i (y_i \partial_\tau x_i - x_i \partial_\tau y_i) + 2\sum A(x_i^2 + y_i^2)\} + i\lambda$$

Eq. of motion

Real eqn.
$$\begin{cases} \partial_\tau(\partial_\tau x_i - 2Ay_i) - (m^2 x_i + 2A\partial_\tau y_i + 2A^2 x_i) = \\ \partial_\tau(\partial_\tau y_i + 2Ax_i) - (m^2 y_i - 2A\partial_\tau x_i + 2A^2 y_i) = \end{cases}$$

$$-2\sum_i (y_i \partial_\tau x_i - x_i \partial_\tau y_i) + 2\sum_i A(x_i^2 + y_i^2) + i\lambda = 0$$

imaginary

$$A = \frac{i\lambda + 2\sum_i (x_i^2 + y_i^2)}{\sum_i (y_i \partial_\tau x_i - x_i \partial_\tau y_i)}$$

complex in general.

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