Probability and inference practice exercises II

(*not* to be handed in or graded)

B.1 Suppose you have a *fair* six-sided die. (*Fair* means that when the die is thrown it is equally probable that any of the six sides will land face up.) Now suppose that you throw the die three times. What is the probability that you will get:

(a) a 1 on the first throw, a 2 on the second throw, and a 3 on the third throw?

(b) a 1, a 2, and a 3 in any order?

(c) three sixes?

(d) one 4 and two 5s, in any order?

B.2 What is the probability of getting the sequence *hhtt* in four tosses of a fair coin? (That is, two heads, *followed by* two tails.) If you *know* that the first two are heads, what is the probability that the last two are tails? If you don't know anything about the first two tosses, what is the probability that the last two are tails?

B.3 Suppose that we have a bent coin which when tossed has a probability of landing heads up that is different from $\frac{1}{2}$. In particular let us consider a coin for which $P(\text{heads}) = 0.1$. We will call the total number of tosses N, and the total number of heads obtained will be called r.

(a) For $N = 2$ tosses, what is the probability distribution of r? [*Hint for checking your answer*: What is the probability of *hh*, *ht*, *th*, *tt*?]

(b) Now the coin is tossed N=1000 times. *Sketch* the probability distribution of r. Mark on your sketch the *mean* number of heads, and the *standard deviation* of the number of heads.

['The *mean* of r' has identical meaning to 'the *average* of r', and 'the expected value of r'. Common notations for the mean of r include $\bar{r}$, $\mathcal{E}[r]$, and $\langle r \rangle$.]

B.4 A die with faces 1, 2, 3, 4, 5, 6 is rolled. It is a "fair" die, that is, $p_r = 1/6$ for all r.

(a) The die is rolled once. Call the outcome the 'score', r. What is the mean of the score?

(b) Two dice are rolled. We define s to be the sum of the two scores, $s = r_1 + r_2$. What is the mean of s?

(c) What is the probability that s is even?

(d) What is the probability distribution of s? Sketch the distribution. What is the relationship of $P(s)$ to *convolution*? For general distributions $P(r_1)$ and $P(r_2)$, what is the formula for $P(s)$? If you don't know the meaning of *convolution*, look it up. You may have used the function `conv` in octave in Alan Beardon's course. This is the convolution function.

(e) Let m be the **larger** of the two scores, $m = \max(r_1, r_2)$. What is the probability distribution of m? What is the probability that m is even?

B.5 Box 1 contains 1 red ball and 3 black balls. Box 2 contains 1 red ball, 1 white ball, and 1 black ball. Box 3 contains 1 red ball and 1 black ball.

(a) A box is chosen at random and one ball is drawn. What is the probability that the ball is red?

(b) Given that the ball is red, what are the probabilities that the chosen box was box k, for $k = 1, 2, 3$?

B.6 Joseph and Mary have identical sets of four cards. Each set contains a Jack, Queen, King, and Ace. The packs are shuffled and then Joseph and Mary reveal their cards, one card at a time, at the same time as each other. If their two cards are identical, they shout "snap!"

(a) What is the probability of getting a "snap" on the first turn?

(b) When they have revealed all four cards, what is the mean number of times that they will have said "snap"?

(c) What is the probability that they will say "snap" four times?

B.7 Suppose that it is known that 1% of the population have a nasty disease. There is a test for this disease; the outcome of the test is either 'positive' or 'negative', and the test is known to be 98% reliable. That is, the test outcome is positive in 98% of people who have the disease, and the outcome is positive in 2% of the people who do not have it. Joe takes the test, and the outcome is positive. What is the probability that Joe has got the disease?

F.8 The random variable u has a uniform probability density $P(u) = 1$, for $u \in (0, 1)$. Sketch the probability densities of the quantities $y = u^2$, $z = \sqrt{u}$, and $l = \log(u)$.

F.9 On a single graph, sketch a Gaussian probability density with mean 2 and standard deviation 2, and a Gaussian with mean 1 and standard deviation 1. Indicate the approximate probability density at the mean of each density.

G.10 Three identical skipping ropes are dropped in a heap. The six free ends of the ropes are picked up by three people – each person picks up one free end with his right hand and another free end with his left. Then they all step away from each other.

(a) What is the probability that every person finds he is holding a single skipping rope?

(b) What is the probability that the three people are joined in a single closed chain?