An Introduction to Quantum Physics
and
Relativistic Quantum Field Theory

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Tutorials: part 3

Noether Theorem, Symmetries and Special Relativity

A. Noether Theorem and Symmetries

1. Consider a system composed of N nonrelativistic particles of masses m_α ($\alpha = 1, 2, \cdots, N$) and position vectors $\vec{r}_\alpha(t)$ with respect to some inertial frame, subjected to a collection of conservative forces of which the total potential energy, $V(|\vec{r}_\alpha - \vec{r}_\beta|)$, is a function only of the pairwise distances between these particles.

   a. Using the method indicated on page 106 of the Lecture notes (Part 1), identify the Noether charges associated to the invariance of this system under constant time translations, space translations and space rotations.

   b. Moving to the Hamiltonian formulation of this dynamics, establish the algebra of Poisson brackets of all these quantities.

   c. Check explicitly that all these Noether charges indeed generate on phase space the infinitesimal symmetry transformations of which they are the conserved charges.

2. Consider a system with as single degree of freedom a cartesian coordinate $x(t)$ and its conjugate momentum $p(t)$. To check that p is the generator of constant translations in x at the quantum level, using the Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar$, show that $\hat{p}$ is the generator of infinitesimal x translations, and that its exponentiated action, $e^{\frac{i}{\hbar}a\hat{p}}$, a being a constant in x space, indeed generates finite translations in x of the quantum states.

3. Using whatever approach, determine the Noether charges associated to the invariance of the free nonrelativistic particle under Galilei boosts, and compute the algebra of their Poisson brackets as well as their action on the phase space variables.

4. Using whatever approach, determine the Noether charges associated to the invariance of the Landau problem, extended with a static homogeneous electric field within the plane, under space rotations and translations, and compute the algebra of their Poisson brackets as well as their action on the phase space variables.

5. Consider the following Lagrangian,

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}_{1i}^2 + \dot{x}_{2i}^2\right) + \frac{1}{2}B\epsilon_{ij}\left(x_{1i}\dot{x}_{1j} - x_{2i}\dot{x}_{2j}\right) - \frac{1}{2}k_0\left(x_{2i} - x_{1i}\right)^2 \\ &= \frac{1}{2}(2m)\dot{X}_i^2 + \frac{1}{2}\left(\frac{1}{2}m\right)\dot{u}_i^2 + \frac{1}{2}B\epsilon_{ij}\left(\dot{X}_i u_j - X_i\dot{u}_j\right) - \frac{1}{2}k_0 u_i^2, \end{aligned}$$

describing a system of two nonrelativistic particles of identical mass m moving in a two dimensional euclidean plane, being coupled with opposite electric charges to a static and homogeneous magnetic field perpendicular to that plane (the choice of vector potential is in the symmetric gauge; for the sake of the Exercise, the Coulomb interaction between the two charges is ignored). The value B absorbs the charge of the particle of position vector $\vec{r}_1$ of which the cartesian coordinates are x_{1i} $(i = 1, 2)$, while x_{2i} are the cartesian coordinates of the second particle of position vector $\vec{r}_2$. The second expression for the Lagrangian is in terms of the center-of-mass and relative coordinates of the system, $X_i = \frac{1}{2}(x_{2i} + x_{1i})$ and $u_i = x_{2i} - x_{1i}$, respectively.

By considering the manner in which the Lagrangian transforms, determine the Noether charges associated to the invariance of the system under space rotations and translations, and compute the algebra of their Poisson brackets as well as their action on the phase space variables.

6. **Dynamical symmetries**.
There are examples of systems of which the Hamiltonian formulation possesses more symmetries than their Lagrangian formulation, so-called dynamical symmetries. One famous example is the Kepler or Coulomb problem with a potential energy in $1/r$ in whatever space dimension (see the next Problem). Another is the spherically symmetric harmonic oscillator in whatever space dimension. Let us restrict to two euclidean space dimensions and consider the dynamics of the spherically symmetric harmonic oscillator in the plane. An obvious space symmetry is that of SO(2)=U(1) rotations in the plane, with as unique conserved charge the oscillator's angular-momentum. However that single property cannot explain why all classical trajectories are closed (they are all ellipses centered onto the origin; this result is also to be established if it is not obvious). By considering the Hamiltonian first-order action of the system, and combining the coordinates and their conjugate momenta into complex variables associated both to rotations in the plane and the creation and annihilation Fock operators of the quantum oscillator, identify a larger SU(2) symmetry acting on phase space. By applying Noether's theorem to this first-order Hamiltonian action, identify the corresponding Noether charges and determine their algebra of Poisson brackets.

7. **Dynamical symmetries: A purely algebraic solution for the hydrogen atom**.
Let us consider a particle of (reduced) mass m and position vector $\vec{r}(t) = \{x_i(t);\ i = 1, 2, 3\}$ in three euclidean space subjected to the Coulomb-Kepler central potential, with Lagrange function
$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(r), \qquad V(r) = -\frac{\lambda}{r}, \qquad r = |\vec{r}\,|,$$
λ being a constant setting the strength of the central force. For positive (resp., negative) λ, this force is attractive (resp., repulsive). In the case of the Kepler problem for two massive bodies of masses m_1 and m_2 one has $\lambda = G_N m_1 m_2$ (in which case $m = m_1 m_2/(m_1+m_2)$), while for the Coulomb problem for two electric charges Q_1 and Q_2 (in S.I. units) one has $\lambda = -Q_1 Q_2/(4\pi\epsilon_0)$. Note that for a hydrogenoid atom composed of a single electron of charge $-|e|$ and a point nucleus of total charge $Z|e| > 0$, one has
$$\lambda = \frac{Ze^2}{4\pi\epsilon_0} = \hbar c\, Z\alpha, \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137},$$
α being the atomic fine structure constant.

The Hamiltonian formulation of this dynamics is thus specified by the Hamiltonian
$$H = \frac{1}{2}\vec{p}^{\,2} + V(r) = \frac{1}{2m}\vec{p}^{\,2} - \frac{\lambda}{r},$$

with the canonical brackets for the conjugate variables (x_i, p_i), $\{x_i, p_j\} = \delta_{ij}$ $(i, j = 1, 2, 3)$. It is well known that besides the total energy, H, and orbital angular-momentum, $\vec{L} = \vec{r} \times \vec{p}$, namely $L_i = \epsilon_{ijk} x_j p_k = -\epsilon_{ijk} p_j x_k$, which are conserved quantities for this system, for this particular choice of interaction potential there exists a second conserved vector quantity, namely the Laplace–Runge–Lenz vector,

$$\vec{A} = \vec{p} \times \vec{L} - \lambda m \hat{r}, \qquad A_i = \epsilon_{ijk} p_j L_k - \lambda m \frac{x_i}{r} = -\epsilon_{ijk} L_j p_k - \lambda m \frac{x_i}{r} = \left\{ \frac{1}{2} L^2 - \lambda m r, p_i \right\},$$

where $L^2 \equiv \vec{L}^2 \equiv L_i^2$.

The purpose of the present analysis is to extend to the quantum dynamics the existence of these conserved quantities, and even manage to compute the spectrum of negative energy bound states in the attractive case using purely algebraic considerations. In particular the existence of the Laplace–Runge–Lenz vector is related to a dynamical symmetry in this system, which accounts for the degeneracies of the energy spectrum of the hydrogen atom, for instance. In three euclidean space it turns out the dynamical symmetry extends the SO(3) rotational symmetry to a SO(4) symmetry for the bound states. A similar result generalises to an euclidean space of any dimension d with then the rotational symmetry SO(d) extended into a dynamical SO($d+1$) symmetry for the bound states.

The quantised system is defined by the Heisenberg algebra commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij},$$

while the above classically conserved quantities have the following quantum operator counterparts,

$$\hat{H} = \frac{1}{2m} \hat{p}_i^2 - \frac{\lambda}{\hat{r}}, \qquad \hat{L}_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k = -\epsilon_{ijk} \hat{p}_j \hat{x}_k, \qquad \hat{A}_i = \frac{i}{\hbar} \left[\hat{p}_i, \frac{1}{2} \hat{L}^2 - \lambda m \hat{r} \right],$$

where $\hat{r} = \sqrt{\hat{x}_i^2}$ and $\hat{L}^2 \equiv \hat{L}_i^2$. Note that these choices all define hermitian and self-adjoint operators.

a. Using $[\hat{p}_i, \hat{r}] = -i\hbar \hat{x}_i / \hat{r}$, establish the following alternative expressions

$$\begin{aligned} \hat{A}_i &= \frac{1}{2} \left(\epsilon_{ijk} \hat{p}_j \hat{L}_k - \epsilon_{ijk} \hat{L}_j \hat{p}_k \right) - \lambda m \frac{\hat{x}_i}{\hat{r}} \\ &= \epsilon_{ijk} \hat{p}_j \hat{L}_k - i\hbar \hat{p}_i - \lambda m \frac{\hat{x}_i}{\hat{r}} = -\epsilon_{ijk} \hat{L}_j \hat{p}_k + i\hbar \hat{p}_i - \lambda m \frac{\hat{x}_i}{\hat{r}}, \end{aligned} \tag{1}$$

as well as the useful identities,

$$\hat{x}_i \hat{L}_i = 0 = \hat{L}_i \hat{x}_i, \qquad \hat{p}_i \hat{L}_i = 0 = \hat{L}_i \hat{p}_i, \qquad \hat{A}_i \hat{L}_i = 0 = \hat{L}_i \hat{A}_i.$$

b. Considering first the angular-momentum operator, one has the following properties under infinitesimal spatial rotations,

$$\left[\hat{L}_i, \hat{x}_j \right] = i\hbar \epsilon_{ijk} \hat{x}_k = \left[\hat{x}_i, \hat{L}_j \right], \qquad \left[\hat{L}_i, \hat{p}_j \right] = i\hbar \epsilon_{ijk} \hat{p}_k = \left[\hat{p}_i, \hat{L}_j \right],$$

hence,

$$\left[\hat{L}_i, \hat{x}_j^2 \right] = 0 = \left[\hat{L}_i, \hat{p}_j^2 \right],$$

as well as,

$$\left[\hat{L}_i, \hat{L}_j \right] = i\hbar \epsilon_{ijk} \hat{L}_k.$$

As a consequence, show that $\hat{H}$ and $\hat{L}_i$ are indeed conserved quantities, namely

$$\left[\hat{H}, \hat{H} \right] = 0 = \left[\hat{L}_i, \hat{H} \right].$$

c. Turning now to the Laplace–Runge–Lenz vector, using one of its expanded expressions above (not in terms of the commutator definition), show that this quantum operator is also conserved,

$$\left[\hat{A}_i, \hat{H}\right] = 0.$$

Exploiting the expression of $\hat{A}_i$ in terms of a commutator and applying the Jacobi identity, establish that under spatial rotations these operators indeed define the components of a vector quantity,

$$\left[\hat{L}_i, \hat{A}_j\right] = i\hbar\epsilon_{ijk}\hat{A}_k = \left[\hat{A}_i, \hat{L}_j\right].$$

d. Using one of the expressions in (1) above, as well as $[\hat{p}_i, 1/\hat{r}] = i\hbar\hat{x}_i/\hat{r}^3$, through a careful and patient calculation, check that the Laplace–Runge–Lenz operator obeys the algebra,

$$\left[\hat{A}_i, \hat{A}_j\right] = -i\hbar\epsilon_{ijk}\left(2m\hat{H}\right)\hat{L}_k.$$

It thus proves useful to introduce the following normalised operators,

$$\hat{D}_i = \frac{1}{\sqrt{2m|\hat{H}|}}\hat{A}_i,$$

which are well defined since $\hat{H}$ commutes with both $\hat{L}_i$ and $\hat{A}_i$, and which are such that

$$\left[\hat{D}_i, \hat{D}_j\right] = -i\hbar\, s\, \epsilon_{ijk}\hat{L}_k, \qquad s = \mathrm{sgn}\, \hat{H}.$$

The operators $\hat{L}_i$ and $\hat{D}_i$ thus form an algebra which closes onto itself. For $s = -1$, namely bound states, this algebra is that of SU(2)×SU(2), namely SO(4), as will now be established. For $s = +1$, namely unbound states, the algebra is that of the noncompact group SO(1,3), which shall not be considered here.

e. When $s = -1$, consider the collection of operators $\hat{L}_{\alpha\beta}$ with $\alpha, \beta = 1, 2, 3, 4$ such that $\hat{L}_{\beta\alpha} = -\hat{L}_{\alpha\beta}$ and defined by

$$\hat{L}_{ij} = \epsilon_{ijk}\hat{L}_k, \qquad \hat{L}_{4i} = \hat{D}_i, \qquad i, j, k = 1, 2, 3.$$

Check that their algebra is

$$\left[\hat{L}_{\alpha\beta}, \hat{L}_{\gamma\delta}\right] = i\hbar\left(\delta_{\alpha\gamma}\hat{L}_{\beta\delta} - \delta_{\alpha\delta}\hat{L}_{\beta\gamma} - \delta_{\beta\gamma}\hat{L}_{\alpha\delta} + \delta_{\beta\delta}\hat{L}_{\alpha\gamma}\right),$$

which is in fact the SO(4) algebra in four dimension euclidean space. As a matter of fact the SO(4) algebra is isomorphic to the SU(2)× SU(2) algebra, as may be seen as follows. Defining the operators

$$\hat{R}_i = \frac{1}{2}\left(\hat{L}_i + \hat{D}_i\right), \qquad \hat{S}_i = \frac{1}{2}\left(\hat{L}_i - \hat{D}_i\right).$$

one easily finds these span two commuting SU(2) algebras,

$$\left[\hat{R}_i, \hat{R}_j\right] = i\hbar\epsilon_{ijk}\hat{R}_k, \qquad \left[\hat{S}_i, \hat{S}_j\right] = i\hbar\epsilon_{ijk}\hat{S}_k, \qquad \left[\hat{R}_i, \hat{S}_j\right] = 0,$$

each having its own quadratic Casimir operator, $\hat{R}^2 \equiv \hat{R}_i^2$ and $\hat{S}^2 \equiv \hat{S}_i^2$,

$$\left[\hat{R}_i, \hat{R}^2\right] = 0 = \left[\hat{S}_i, \hat{S}^2\right].$$

However, these two Casimir operators are not independent in the present system. Using the fact that $\hat{L}_i\hat{A}_i = 0 = \hat{A}_i\hat{L}_i$, show that

$$\hat{R}^2 - \hat{S}^2 = 0.$$

Consequently, bound quantum states may be characterised in terms of SU(2)×SU(2) quantum numbers related to representations of spin j_+, say for the $\hat{R}_i$ algebra, and of spin j_- for the $\hat{S}_i$ algebra, with however $j_+ = j_- = j$ and j being a positive integer or half-integer number.

f. Using the identity

$$\hat{A}_i + \lambda m \frac{\hat{x}_i}{\hat{r}} = \epsilon_{ijk}\hat{p}_j\hat{L}_k - i\hbar\hat{p}_i = -\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i,$$

from which follows the relation

$$\left(\hat{A}_i + \lambda m \frac{\hat{x}_i}{\hat{r}}\right)^2 = \left(\epsilon_{ijk}\hat{p}_j\hat{L}_k - i\hbar\hat{p}_i\right)\left(-\epsilon_{i\ell m}\hat{L}_\ell\hat{p}_m + i\hbar\hat{p}_i\right),$$

show that one has

$$\hat{A}_i^2 = 2m\hat{H}\left(\hat{L}^2 + \hbar^2\right) + (\lambda m)^2,$$

leading to,

$$\hat{H} = -\frac{1}{2}\frac{m\lambda^2}{\hbar^2}\frac{1}{1 + \frac{1}{\hbar^2}\left(\hat{L}^2 - s\hat{D}^2\right)},$$

with in particular for bound states, $s = -1$,

$$\hat{H} = -\frac{m\lambda^2}{2\hbar^2}\frac{1}{1 + \frac{2}{\hbar^2}\left(\hat{R}^2 + \hat{S}^2\right)}.$$

Conclude that the energy spectrum of the system is given by

$$E_j = -\frac{m\lambda^2}{2\hbar^2}\frac{1}{(2j+1)^2}, \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$$

each such level having a degeneracy $(2j+1)^2$. This result indeed explains the degenerate spectrum of the hydrogen atom, with n^2 states at level $E_n = -\frac{1}{2}(Z\alpha)^2 mc^2 \frac{1}{n^2}$ and $n = 1, 2, 3, \ldots$.

B. Special Relativity

1. In its proper inertial frame R' a stick of length L_0 is positioned in the plane $(x'y')$ with an angle θ_0 with respect to the axis $x' > 0$. The frame R' is itself in uniform rectilinear motion with velocity $v_0 > 0$ with respect to the axis $x > 0$ of an inertial frame R of which all coordinates axes (xyz) coincide with those $(x'y'z')$ of the frame R'.

   a. In the frame R what is the value of the angle θ of the stick with the axis $x > 0$? (One sets of course $\gamma_0 = 1/\sqrt{1-\beta_0^2}$ with $\beta_0 = v_0/c$).

   b. What is the stick's length measured in the frame?

2. **Composition of Lorentz boosts and the velocity addition theorem**.

   a. Let us consider a first Lorentz boost in the $x > 0$ direction of normalised velocity $\beta_1 = v_1/c$, followed by a second one still in the direction $x > 0$ of normalised velocity $\beta_2 = v_2/c$. Show that the result is equivalent to a single Lorentz boost in the direction $x > 0$ of normalised velocity β_{21} given by (note this is the addition theorem for velocities)
$$\beta_{21} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}.$$

   b. In order to check that the speed of light may never be reached through successive Lorentz boosts, let us consider a succession of identical Lorentz boosts all in the direction $x > 0$ and of normalised velocity $\beta = v/c$, v being a given velocity. These Lorentz boosts are applied to an inertial frame initially observed at rest, of normalised velocity $\beta_0 = 0$. After n such Lorentz boosts the frame has a normalised velocity β_n. Establish the recursion relation for successive values of β_n.

   c. In order to solve these recursion relations, it proves useful to introduce a hyperbolic function parametrisation of the normalised velocities (namely exploit the hyperbolic character of the Minkowski geometry of spacetime), in terms of the rapidity ω,
$$\cosh\omega = \gamma, \quad \sinh\omega = \beta\gamma, \quad \tanh\omega = \beta, \quad \beta = \frac{v}{c}, \quad \gamma = (1-\beta^2)^{-1/2}.$$
   Show then that the solution to the recursion relation is
$$\beta_n = \tanh\left(n \,\mathrm{arctanh}\, \frac{v}{c}\right).$$

   d. Based on this result, study the behaviour of the asymptotic velocity, $\lim_{n\to\infty} \beta_n$.

3. **Lorentz transformations and velocity compositions**.
   Consider two inertial frames R and R' sharing the same right-handed orthonormalised basis vectors $\{\hat{I}_1, \hat{I}_2, \hat{I}_3\}$, and of which the origins coincide at time $t = 0 = t'$. The spacetime coordinates of a same event are denoted (ct, x, y, z) with respect to the frame R and (ct', x', y', z') with respect to R'. The frame R' moves with respect to R with the constant velocity $\vec{v}_0 = v_0 \hat{I}_1$, with $v_0 > 0$. We set $\beta_0 = v_0/c$ and $\gamma_0 = (1-\beta_0^2)^{-1/2}$.

   With respect to R, a photon of frequency ν propagates in the direction $\hat{n}$ characterised by angles (θ, φ) such that $\hat{n} = \cos\theta\,\hat{I}_1 + \cos\varphi\sin\theta\,\hat{I}_2 + \sin\varphi\sin\theta\,\hat{I}_3$. One wishes to determine the characteristics of the trajectory of this photon as observed from R'.

   a. Given the expressions for the relevant Lorentz boost, establish the addition theorem for velocities as measured with respect to the two frames R and R'.

b. Applying this result to the photon, show that its velocity in R' is of norm c, as it should.

c. Show that with respect to the given basis vectors and the frame R' the angles (θ', φ') defined in a likewise way to the one above for the trajectory of the photon are

$$\cos\theta' = \frac{\cos\theta - \beta_0}{1-\beta_0\cos\theta}, \qquad \sin\theta' = \frac{1}{\gamma_0}\frac{\sin\theta}{1-\beta_0\cos\theta}, \qquad \tan\theta' = \frac{1}{\gamma_0}\frac{\sin\theta}{\cos\theta-\beta_0}, \qquad \varphi' = \varphi.$$

d. Given that for the considered photon $E = h\nu$ and $\vec{\beta} = \vec{p}c/E$, with $(E, \vec{p}c)$ defining the components of a four-vector, show that the frequency ν' of the photon in the frame R' is (**the electromagnetic Doppler effect**)

$$\nu' = \nu\,\frac{1-\beta_0\cos\theta}{\sqrt{1-\beta_0^2}} = \nu\sqrt{\frac{(1-\beta_0\cos\theta)^2}{(1-\beta_0)(1+\beta_0)}}.$$

4. Given a unit of mass m_0, the relativistic energy and momentum of a free particle are measured to be

$$E = 5\,m_0c^2, \qquad \vec{p} = 4\,m_0c\,\hat{x},$$

$\hat{x}$ being a normalised direction in space.

a. What is the mass m_1 of the particle?

b. What is the norm v of the velocity of the particle?

c. If τ_{proper} is the proper lifetime of the particle, what is its lifetime τ in the inertial frame in which it is observed with the above values of E and $\vec{p}$?

d. The particle of mass m_1 suddenly collides in an elastic scattering with another particle of unknown mass m_2 initially at rest. After the collision the latter particle is observed in the $\hat{x}$ direction with a velocity of norm $v_2 = \frac{3}{5}\,c$. What is the value of its mass m_2?

e. After the collision, what is the velocity $\vec{v}_1$ of the incoming particle?

5. Let us consider the disintegration

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu$$

with the mass values $m_{\pi^+}c^2 = 139{,}57$ MeV, $m_{\mu^+}c^2 = 105{,}66$ MeV and $m_{\nu_\mu}c^2 = 0$ MeV, the initial particle being at rest.

a. What is the neutrino's energy E_{ν_μ} in the final state?

b. What is the μ^+'s energy E_{μ^+} in the final state?

c. In units of c, what is the velocity of the μ^+?

d. In units of c, what is the velocity of the neutrino ν_μ?

6. **Compton scattering**.
Consider the Compton scattering of a photon on an electron initially at rest in the inertial frame of the laboratory,

$$\gamma + e^- \longrightarrow \gamma + e^-.$$

Let E_γ and $\vec{p}_\gamma$ be the energy and momentum of the incoming photon, and E'_γ and $\vec{p}_\gamma\,'$ be those of the photon in the final state scattered in the angular direction θ with respect to the incoming photon. The electron mass is denoted m_e.

a. Show that as a function of the scattering angle the final photon energy is

$$E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{m_e c^2}(1-\cos\theta)}.$$

b. Using the quantum relation between the energy and the wavelength of a photon in vacuum, establish the Compton effect

$$\lambda'_\gamma - \lambda_\gamma = \lambda_e\,(1-\cos\theta)\,,$$

λ'_γ and λ_γ being the initial and final photon wavelengths, respectively, and λ_e the Compton wavelength of the electron

$$\lambda_e = \frac{2\pi\hbar c}{m_e c^2} = \frac{hc}{m_e c^2}.$$

7. Let us consider the relativistic equation of motion of a particle of mass m subjected to a constant force $\vec{F} = F\,\hat{F}$,

$$\frac{d\vec{p}(t)}{dt} = \vec{F}, \qquad \vec{p}(t) = \frac{m\vec{v}(t)}{\sqrt{1-\frac{\vec{v}^{\,2}(t)}{c^2}}},$$

given the initial conditions $\vec{x}(t=0)=\vec{x}_0$ and $\vec{p}(t=0)=\vec{p}_0$. In the nonrelativistic limit of Newton's mechanics the trajectory is of constant acceleration and generally parabolic (unless the initial data $\vec{x}_0$ and $\vec{p}_0$ are colinear, or one at least is vanishing, in which case the trajectory is linear and of constant acceleration of course). What is the situation in the relativistic context?

a. Show that the solution for the velocity is

$$\vec{v}(t) = \frac{1}{m}\frac{\vec{F}t+\vec{p}_0}{\sqrt{1+\frac{1}{m^2c^2}\left(\vec{F}t+\vec{p}_0\right)^2}}.$$

How does the norm of this velocity evolve in time?

b. Given the particular initial conditions $\vec{p}_0=\vec{0}$ and $\vec{x}_0=\vec{0}$, show that the trajectory is given as

$$\vec{v}(t) = \frac{1}{m}\frac{\vec{F}t}{\sqrt{1+\frac{F^2t^2}{m^2c^2}}}, \qquad \vec{x}(t) = \frac{mc^2}{F}\,\hat{F}\left[\sqrt{1+\frac{F^2t^2}{m^2c^2}}-1\right].$$

Consider the graphs of $|\vec{v}(t)|$ and $|\vec{x}(t)|$ in order to observe the transition from the nonrelativistic regime to the relativistic one.

8. Show that the covariant equation of motion of a massive particle in a background electromagnetic field, namely

$$mc^2\,\frac{du_\mu}{ds} = qc\,F_{\mu\nu}\,u^\nu, \quad ds = cd\tau, \quad u^\mu = \frac{dx^\mu}{ds},$$

with τ being the proper time, is equivalent to the Lorentz force equation.