An Introduction to Quantum Physics
and
Relativistic Quantum Field Theory

AIMS Lectures: 24 January – 11 February 2011
Tutorials: part 3 – Examples of Solutions

Noether Theorem, Symmetries and Special Relativity

A. Noether Theorem and Symmetries

1. Consider a system composed of N nonrelativistic particles of masses m_α $(\alpha = 1, 2, \cdots, N)$ and position vectors $\vec{r}_\alpha(t)$ with respect to some inertial frame, subjected to a collection of conservative forces of which the total potential energy, $V(|\vec{r}_\alpha - \vec{r}_\beta|)$, is a function only of the pairwise distances between these particles.

   a. Using the method indicated on page 106 of the Lecture notes (Part 1), identify the Noether charges associated to the invariance of this system under constant time translations, space translations and space rotations.

   b. Moving to the Hamiltonian formulation of this dynamics, establish the algebra of Poisson brackets of all these quantities.

   c. Check explicitly that all these Noether charges indeed generate on phase space the infinitesimal symmetry transformations of which they are the conserved charges.

See the Lecture notes (Part 1), Sections 5.5.1. and 5.5.2., as well as the argument at the end of Section 5.4.

The advocated method for identifying the Noether charges consists in replacing, momentarily, the constant parameters of the symmetry transformation by arbitrary functions of (space[1])-time. The linearised variation of the action, which for constant parameters is invariant up to a total time derivative, must then be expressible in terms of a linear combination of the (space)time derivatives of the symmetry parameters, up to a total time derivative. The coefficients of this linear combination must correspond to the Noether charges (possibly up to a sign as compared to the general formula derived in the Noether theorem). The reason for this last fact is that upon integration by parts, one obtains for the linearised variation of the action, again up to some total time derivative, a linear combination of the symmetry parameters of which the coefficients are the time derivatives of the would-be Noether charges. However since when the symmetry parameters are constant the linearised variation of the action must simply reduce to a total time

[1]In the case of a field theory.

derivative (which could be vanishing) it follows that the time derivatives of the would-be Noether charges must vanish. In other words these are conserved quantities which must correspond to the Noether charges associated to the symmetry transformation. The purpose of the present Problem is to apply this method to any system of nonrelativistic particles in interaction through a potential which is invariant under space translations and rotations.

a. Such a system is described by the Lagrange function

$$L = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}_\alpha}^{\,2} \; - \; V\left(|\vec{r}_\alpha - \vec{r}_\beta|\right), \tag{1}$$

in notations that should be self-explanatory and which are those of the Lecture notes. Let us first consider the invariance of this dynamics under constant translations in time,

$$t'(t) = t + t_0, \qquad \vec{r}_\alpha{}'(t') = \vec{r}_\alpha(t). \tag{2}$$

It should be obvious that such a transformation leaves the Lagrange function invariant without inducing a total time derivative. Given the principle of the advocated method to be used, let us now momentarily take the symmetry parameter t_0 to be a time dependent function, $t_0(t)$, and linearise the variation of the action in that function and its time derivative. Since there is a nontrivial Jacobian factor arising when $t_0(t)$ is a function of t, we must consider the action itself and its variation. We have

$$\frac{dt'}{dt} = 1 + \frac{dt_0(t)}{dt}, \qquad \frac{dt}{dt'} = \frac{1}{1 + \frac{dt_0(t)}{dt}} \simeq 1 - \frac{dt_0(t)}{dt}, \tag{3}$$

where in the second equality we have already expanded to first order in $t_0(t)$ using the series $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$. We may now linearise the variation of the action

$$S[\vec{r}_\alpha{}'] \; - \; S[\vec{r}_\alpha] = \int dt' \, L\left(\vec{r}_\alpha{}', \frac{d\vec{r}_\alpha{}'}{dt'}\right) \; - \; \int dt \, L\left(\vec{r}_\alpha, \frac{d\vec{r}_\alpha}{dt}\right). \tag{4}$$

We thus have to first order in $t_0(t)$,

$$\delta S = \int dt \left\{ \left[1 + \frac{dt_0(t)}{dt}\right] L\left(\vec{r}_\alpha, \left[1 - \frac{dt_0(t)}{dt}\right] \frac{d\vec{r}_\alpha}{dt}\right) \; - \; L\left(\vec{r}_\alpha, \frac{d\vec{r}_\alpha}{dt}\right) \right\}, \tag{5}$$

hence

$$\delta S = \int dt \, \frac{dt_0(t)}{dt} \left\{ \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}_\alpha}^{\,2} \; - \; V\left(|\vec{r}_\alpha - \vec{r}_\beta|\right) \; - \; \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}_\alpha}^{\,2} \right\}, \tag{6}$$

since $(1 - dt_0/dt)^2 \simeq 1 - 2dt_0/dt$, and finally

$$\delta S = \int dt \, \frac{dt_0(t)}{dt} \left\{ - \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}_\alpha}^{\,2} \; - \; V\left(|\vec{r}_\alpha - \vec{r}_\beta|\right) \right\}. \tag{7}$$

In conclusion the Noether charge associated to time translation invariance is indeed, up to a sign, the total energy, *i.e.*, the Hamiltonian of the system,

$$\sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}_\alpha}^{\,2} \; + \; V\left(|\vec{r}_\alpha - \vec{r}_\beta|\right) = H. \tag{8}$$

Turning now to translations in space

$$t'(t) = t, \qquad \vec{r}_\alpha{}'(t') = \vec{r}_\alpha \; + \; \vec{r}_0, \tag{9}$$

it is clear that these transformations define a global symmetry of which the parameters are the constant components of the translation vector $\vec{r}_0$ common to all position vectors $\vec{r}_\alpha$, since the Lagrange function is then manifestly invariant, without a total derivative term being induced. Replacing now the constant vector by an arbitrary time dependent vector function, $\vec{r}_0(t)$, the linearised variation of the Lagrange function is simply

$$\delta L = \sum_{\alpha=1}^{N} m_\alpha \frac{d\vec{r}_0}{dt} \cdot \dot{\vec{r}}_\alpha = \frac{d\vec{r}_0}{dt} \cdot \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha, \tag{10}$$

since

$$\frac{d\vec{r}_\alpha{}'}{dt'} = \frac{d\vec{r}_\alpha}{dt} + \frac{d\vec{r}_0}{dt}, \qquad \left(\frac{d\vec{r}_\alpha{}'}{dt'}\right)^2 \simeq \dot{\vec{r}}_\alpha{}^2 + 2\frac{d\vec{r}_0}{dt} \cdot \dot{\vec{r}}_\alpha. \tag{11}$$

Consequently, it follows that the Noether charges associated to space translation invariance are the components of the vector quantity

$$\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha = \sum_{\alpha=1}^{N} \vec{p}_\alpha = \vec{P}, \tag{12}$$

which is the total (velocity) momentum of the system.

Finally let us consider rotations in space

$$t'(t) = t, \qquad \vec{r}_\alpha{}'(t') = R \cdot \vec{r}_\alpha(t), \tag{13}$$

where R stands for a 3×3 rotation matrix acting on the components of the position vectors $\vec{r}_\alpha$ all in an identical manner. If these components are denoted as $x^i_\alpha(t)$ with $i = 1, 2, 3$, and considering an infinitesimal or linearised rotation, we may write (the summation over repeated indices is implicit)

$$\left(x'_\alpha\right)^i = x^i_\alpha + \theta_j\, \epsilon^{ijk}\, x^k_\alpha, \tag{14}$$

where θ_j are the rotation angles around each of the coordinate axes $j = 1, 2, 3$. In vector form we thus have

$$\delta\vec{r}_\alpha = \vec{\theta} \times \vec{r}_\alpha. \tag{15}$$

Applying the advocated method for finding the Noether charges the rotation angles become arbitrary time dependent functions, $\theta_j(t)$, in terms of which to linearise again the variation of the Lagrange function which is left invariant, without an induced total derivative term, when the rotation angles are constant parameters. The calculation of δL is identical to the one above for a space translation, in which the quantities $\delta x^i_\alpha = r^i_0$ are replaced by $\delta x^i_\alpha = \epsilon^{ijk}\theta_j x^k_\alpha$. Consequently the associated Noether charges are the components of the vector quantity

$$\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha = \vec{L}, \tag{16}$$

which is the total (orbital) angular-momentum of the system.

b. The above Noether charges are readily expressed in Hamiltonian form,

$$\begin{aligned} H &= \sum_{\alpha=1}^{N} \frac{1}{2m_\alpha} \vec{p}_\alpha{}^2 + V\left(|\vec{r}_\alpha - \vec{r}_\beta|\right), \\ \vec{P} &= \sum_{\alpha=1}^{N} \vec{p}_\alpha, \\ \vec{L} &= \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha. \end{aligned} \tag{17}$$

Knowing the elementary canonical Poisson brackets for the cartesian components of the canonically conjugated variables $\vec{r}_\alpha = (x^i_\alpha)$ and $\vec{p}_\alpha = (p^i_\alpha)$,

$$\left\{x^i_\alpha, p^j_\beta\right\} = \delta^{ij}\,\delta_{\alpha\beta}, \tag{18}$$

a simple calculation readily finds the algebra of Poisson brackets,

$$\begin{aligned} \{H,H\} &= 0, \qquad \left\{H,P^i\right\} = 0, \qquad \left\{H,L^i\right\} = 0, \\ \left\{P^i,P^j\right\} &= 0, \qquad \left\{P^i,L^j\right\} = \epsilon^{ijk}\,P^k, \qquad \left\{L^i,L^j\right\} = \epsilon^{ijk}\,L^k, \end{aligned} \tag{19}$$

where as always the summation over repeated indices is implicit.

c. Let us also consider the variations of the coordinate phase space degrees of freedom generated by these Noether charges. We have

$$\left\{x^i_\alpha, H\right\} = \frac{1}{m_\alpha}p^i_\alpha = \dot{x}^i_\alpha, \quad \left\{x^i_\alpha, P^j\right\} = \delta^{ij}, \quad \left\{x^i_\alpha, L^j\right\} = \epsilon^{ijk}\,x^k_\alpha, \tag{20}$$

so that for infinitesimal symmetry parameters δt_0, $\delta\vec{r}_0$ and $\delta\theta_i$,

$$\left\{x^i_\alpha, \delta t_0\,H\right\} = \delta t_0\,\dot{x}^i_\alpha, \quad \left\{x^i_\alpha, \delta\vec{r}_0\cdot\vec{P}\right\} = \delta r^i_0, \quad \left\{x^i_\alpha, \delta\vec{\theta}\cdot\vec{L}\right\} = \left(\delta\vec{\theta}\times\vec{r}_\alpha\right)^i, \tag{21}$$

which are indeed recognized to coincide with the corresponding infinitesimal symmetry transformations.

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2. Consider a system with as single degree of freedom a cartesian coordinate $x(t)$ and its conjugate momentum $p(t)$. To check that p is the generator of constant translations in x at the quantum level, using the Heisenberg algebra $[\hat{x},\hat{p}] = i\hbar$, show that $\hat{p}$ is the generator of infinitesimal x translations, and that its exponentiated action, $e^{\frac{i}{\hbar}a\hat{p}}$, a being a constant in x space, indeed generates finite translations in x of the quantum states.

See the Lectures notes (Part 1), Section 5.5.2.

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3. Using whatever approach, determine the Noether charges associated to the invariance of the free nonrelativistic particle under Galilei boosts, and compute the algebra of their Poisson brackets as well as their action on the phase space variables.

See the Lectures notes (Part 1), Section 5.5.3.

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4. Using whatever approach, determine the Noether charges associated to the invariance of the Landau problem, extended with a static homogeneous electric field within the plane, under space rotations and translations, and compute the algebra of their Poisson brackets as well as their action on the phase space variables.

See the Lectures notes (Part 1), Section 5.5.4.

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5. Consider the following Lagrangian,

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}_{1i}^2+\dot{x}_{2i}^2\right)+\frac{1}{2}B\epsilon_{ij}\left(x_{1i}\dot{x}_{1j}-x_{2i}\dot{x}_{2j}\right)-\frac{1}{2}k_0\left(x_{2i}-x_{1i}\right)^2 \\ &= \frac{1}{2}\left(2m\right)\dot{X}_i^2+\frac{1}{2}\left(\frac{1}{2}m\right)\dot{u}_i^2+\frac{1}{2}B\epsilon_{ij}\left(\dot{X}_iu_j-X_i\dot{u}_j\right)-\frac{1}{2}k_0u_i^2, \end{aligned}$$

describing a system of two nonrelativistic particles of identical mass m moving in a two dimensional euclidean plane, being coupled with opposite electric charges to a static and homogeneous magnetic field perpendicular to that plane (the choice of vector potential is in the symmetric gauge; for the sake of the Exercise, the Coulomb interaction between the two charges is ignored). The value B absorbs the charge of the particle of position vector $\vec{r}_1$ of which the cartesian coordinates are x_{1i} ($i=1,2$), while x_{2i} are the cartesian coordinates of the second particle of position vector $\vec{r}_2$. The second expression for the Lagrangian is in terms of the center-of-mass and relative coordinates of the system, $X_i=\frac{1}{2}(x_{2i}+x_{1i})$ and $u_i=x_{2i}-x_{1i}$, respectively.

By considering the manner in which the Lagrangian transforms, determine the Noether charges associated to the invariance of the system under space rotations and translations, and compute the algebra of their Poisson brackets as well as their action on the phase space variables.

Let us first consider transformations of the degrees of freedom of the system under space rotations, in infinitesimal form. Denoting by θ the infinitesimal rotation angle in the plane, and using the notations of the Lectures notes discussing the first Noether theorem (see Eq.(390), p. 102, of Part 1 of these notes), such transformations correspond to the variations,

$$\delta t=0,\qquad \delta x_{1i}=-\theta\epsilon_{ij}x_{1j},\qquad \delta x_{2i}=-\theta\epsilon_{ij}x_{2j},\qquad \delta\Lambda=0, \tag{1}$$

namely in terms of the center-of-mass and relative position variables,

$$\delta t=0,\qquad \delta X_i=-\theta\epsilon_{ij}X_j,\qquad \delta u_i=-\theta\epsilon_{ij}u_j,\qquad \delta\Lambda=0. \tag{2}$$

Because of the antisymmetry properties of the symbol ϵ_{ij} in its two indices, it is readily checked that indeed (in this linearised form) the Lagrangian is left invariant under these transformations. In other words, rotations in the plane of constant rotation angle θ define a symmetry of the system.

Consequenly, given the general expression for Noether charges (see Eq.(392), p. 102, of Part 1 of the notes), for spatial rotations the Noether charge is given as,

$$L=-\frac{\partial L}{\partial\dot{X}_i}\epsilon_{ij}X_j-\frac{\partial L}{\partial\dot{u}_i}\epsilon_{ij}u_j=\epsilon_{ij}X_i\Pi_j+\epsilon_{ij}u_ip_j, \tag{3}$$

where Π_i and p_i stand for the conjugate momenta of the configuration space coordinates X_i and u_i, respectively, with the usual canonical Poisson brackets. Note how this expression is of the orbital angular-momentum form, as it should since the Noether charge related to invariance under rotations is to be identified with the total angular-momentum of the system, inclusive in the present case of the magnetic field contribution (since the magnetic field also carries a contribution to the total angular-momentum of the system).

For what translations in the plane are concerned, in infinitesimal form these are given as, once again in the notations of the Lecture notes,

$$\delta t = 0, \qquad \delta x_{1i} = a_i, \qquad \delta x_{2i} = a_i, \qquad \delta\Lambda = -\frac{1}{2}B\epsilon_{ij}a_i u_j, \tag{4}$$

namely,

$$\delta t = 0, \qquad \delta X_i = a_i, \qquad \delta u_i = 0, \qquad \delta\Lambda = -\frac{1}{2}B\epsilon_{ij}a_i u_j. \tag{5}$$

In these expressions, a_i stand for the components of the two dimensional translation vector in the plane. Indeed, the Lagrangian is invariant only up to a total time derivative term under spatial translations, hence a nonvanishing variation for that surface term, $\delta\Lambda$. Once again these transformation properties of the Lagrangian are readily checked.

Consequently, the two Noether charges for planar translations are given by the following two components,

$$P_i = \frac{\partial L}{\partial \dot{X}_j}\delta_{ij} - \left(-\frac{1}{2}B\epsilon_{ij}u_j\right) = \Pi_i + \frac{1}{2}B\epsilon_{ij}u_j, \tag{6}$$

these two components defining a vector quantity in the plane corresponding to the total momentum of the system.

In terms of these expressions, a simple calculation finds for their Poisson brackets,

$$\{L, L\} = 0, \qquad \{P_i, L\} = -\epsilon_{ij}\,P_j, \qquad \{P_i, P_j\} = 0. \tag{7}$$

These are indeed the expected infinitesimal forms, up the associated constant parameter, of the variations of these conserved phase space observables under spatial rotations and translations.

For the basic phase variables, one also finds,

$$\{X_i, L\} = -\epsilon_{ij}X_j, \qquad \{\Pi_i, L\} = -\epsilon_{ij}\Pi_j, \qquad \{u_i, L\} = -\epsilon_{ij}u_j, \qquad \{p_i, L\} = -\epsilon_{ij}p_j, \tag{8}$$

as well as,

$$\{X_i, P_j\} = \delta_{ij}, \qquad \{\Pi_i, P_j\} = 0, \qquad \{u_i, P_j\} = 0, \qquad \{p_i, P_j\} = \frac{1}{2}B\epsilon_{ij}. \tag{9}$$

Note that because of the coupling to the background magnetic field, even though the relative coordinates u_i are invariant under spatial translations, their conjugate momenta are not.

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6. **Dynamical symmetries**.
There are examples of systems of which the Hamiltonian formulation possesses more symmetries than their Lagrangian formulation, so-called dynamical symmetries. One famous example is the Kepler or Coulomb problem with a potential energy in $1/r$ in whatever space dimension (see the next Problem). Another is the spherically symmetric harmonic oscillator in whatever space dimension. Let us restrict to two euclidean space dimensions and consider the dynamics of the spherically symmetric harmonic oscillator in the plane. An obvious space symmetry is that of

SO(2)=U(1) rotations in the plane, with as unique conserved charge the oscillator's angular-momentum. However that single property cannot explain why all classical trajectories are closed (they are all ellipses centered onto the origin; this result is also to be established if it is not obvious). By considering the Hamiltonian first-order action of the system, and combining the coordinates and their conjugate momenta into complex variables associated both to rotations in the plane and the creation and annihilation Fock operators of the quantum oscillator, identify a larger SU(2) symmetry acting on phase space. By applying Noether's theorem to this first-order Hamiltonian action, identify the corresponding Noether charges and determine their algebra of Poisson brackets.

As a brief description of the outline of the solution, the starting point of the analysis is the Hamiltonian first-order action of the system, which may be taken in the following form

$$S[x^i,p_i]=\int dt\left[\frac{1}{2}\left(\dot{x}^1p_1+\dot{x}^2p_2-x^1\dot{p}_1-x^2\dot{p}_2\right)\,-\,\frac{1}{2m}\left(p_1^2+p_2^2\right)-\frac{1}{2}m\omega^2\left((x^1)^2+(x^2)^2\right)\right], \tag{1}$$

where, when compared to the general form $S[q^n,p_n]=\int dt[\dot{q}^np_n-H(q^n,p_n)]$, a partial integration by parts of the first-order term $\dot{q}^np_n$ has been effected to render the expression for the Hamiltonian first-order action as symmetric as possible between the two degrees of freedom $x^i(t)$ $(i=1,2)$ and their conjugate momenta p_i.

Following then the lead of the quantisation of this system discussed in the solutions to the "Tutorials: Part 2", one introduces in succession the following new parametrisations (at the classical level, otherwise at the quantum level there is also an extra factor of $\hbar$ involved in the square root normalisation factors),

$$a_i=\sqrt{\frac{m\omega}{2}}\left[x^i+\frac{i}{m\omega}p_i\right],\qquad i=1,2, \tag{2}$$

$$a_\pm=\frac{1}{\sqrt{2}}\left[a_1\mp ia_2\right],\qquad a_\pm^*=\frac{1}{\sqrt{2}}\left[a_1^*\pm ia_2^*\right]. \tag{3}$$

Substituting these redefinitions in the expression for the above first-order action, one then notices that it may easily be expressed in terms of

$$A=\begin{pmatrix}a_+\\ a_-\end{pmatrix},\qquad A^\dagger=\begin{pmatrix}a_+^* & a_-^*\end{pmatrix}, \tag{4}$$

in the following form

$$S[a_\pm,a_\pm^*]=\int dt\left[-\frac{1}{2}i\left(\dot{A}^\dagger\cdot A\,-\,A^\dagger\cdot\dot{A}\right)\,-\,\omega\,A^\dagger\cdot A\right]. \tag{5}$$

In this form, it is clear that the action is invariant under the following U(2) transformations

$$A\;\longrightarrow A'=U\cdot A, \tag{6}$$

where U is an arbitrary 2×2 complex matrix which is unitary, $U^\dagger=U^{-1}$. Considering then the infinitesimal transformations in U(2), there follows the existence of four Noether charges, one associated to the U(1) part of SU(2) which corresponds to a simple identical phase transformation of both a_+ and a_-, and the remaining three Noether charges being the generators of the actual SU(2) symmetry, for which in addition to being unitary, the matrix U is also of unit determinant, $\det U=1$. It turns out that the U(1) Noether charge is simply

$$Q_0=A^\dagger\cdot A, \tag{7}$$

which is essentially the Hamiltonian of the system up to a factor ω, while the remaining SU(2) charges are

$$Q_a = A^\dagger \cdot T^a \cdot A, \qquad T^a = \frac{1}{2}\sigma^a, \tag{8}$$

σ^a being the three Pauli matrices ($a = 1, 2, 3$). Namely one has

$$Q_1 = \frac{1}{2}\left[a_+^* a_- + a_-^* a_+\right], \quad Q_2 = -\frac{1}{2} i \left[a_+^* a_- - a_-^* a_+\right], \quad Q_3 = \frac{1}{2}\left[a_+^* a_+ - a_-^* a_-\right], \tag{9}$$

such that

$$Q_\pm = Q_1 \pm i Q_2, \quad Q_+ = a_+^* a_-, \quad Q_- = a_-^* a_+. \tag{10}$$

Given that the Poisson brackets of the phase space variables are

$$\left\{a_\pm, a_\pm^*\right\} = -i, \tag{11}$$

a straightforward calculation of the Noether charge algebra finds for the SU(2) part

$$\{Q_i, Q_j\} = \epsilon_{ijk}\, Q_k, \tag{12}$$

while of course the Poisson brackets of H or Q_0 with Q_i ($i = 1, 2, 3$) vanish identically since Q_0 generates the U(1) component of the U(2) symmetry which commutes with the SU(2) component, while on the other hand the Q_i are conserved charges, $\dot{Q}_i = 0$.

It is possible to work out the expressions for these charges in terms of the real phase space variables x^i and p_i. One finds

$$Q_3 = \frac{1}{2}\left[x_1 p_2 - x_2 p_1\right], \tag{13}$$

which is thus the angular-momentum of the system (perpendicular to the plane) up to the factor 1/2. This ought indeed to be the generator of rotations around the axis perpendicular to the plane. On the other hand one also has

$$Q_+ = \frac{m\omega}{4}\left[\left((x^1)^2 + \frac{p_1^2}{m^2\omega^2}\right) - \left((x^2)^2 + \frac{p_2^2}{m^2\omega^2}\right)\right] + i\frac{m\omega}{2}\left(x^1 x^2 + \frac{p_1 p_2}{m^2\omega^2}\right). \tag{14}$$

Separating the real and imaginary parts of the conserved charges $Q_\pm$, and then together with Q_0 or equivalently

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2} m\omega^2 \left((x^1)^2 + (x^2)^2\right), \tag{15}$$

any given solution to the equations of motion is such that these four independent quantities take constant values for the four independent phase space variables (x^i, p_i). However, these different quantities are quadratic expressions in these variables, thus defining conics in phase space. The Hamiltonian defines a 3-ellipse, which thus intersects three other conics of either a parabolic (for Q_2 and Q_3) or an hyperbolic (for Q_1) type. Consequently the intersection of these four conics defines a 1-ellipse in phase space. By projection onto the 2-dimensional configuration space (x^i), necessarily the trajectory in the plane is itself an ellipse, hence also a closed curved. Of course, the fact that the solutions to the Euler–Lagrange equations of motion for $x^i(t)$ are closed trajectories is obvious since they have identical periods $T = 2\pi/\omega$. However the actual reason why this is so finds its very explanation in the existence of the U(2) symmetries of the spherically symmetric oscillator in the plane. Similar considerations apply to the spherically symmetric harmonic oscillator in an euclidean space of arbitrary dimension d. In that general case the dynamical symmetry is U(d). Note also that the identification of these symmetries requires the Hamiltonian formulation. Having identified the Noether charges in phase space, one may apply the Hamiltonian reduction and express the conjugate momenta in terms of the velocities, $p_i = m\dot{x}^i$. However, the resulting infinitesimal variations for x^i and $\dot{x}^i$, even though leaving the Lagrange function invariant, cannot be obtained from a single transformation of the configuration space variables x^i only. Dynamical symmetries are symmetries of the dynamics, namely of its Hamiltonian formulation which in a sense is more fundamental, which are not manifest symmetries of its Lagrangian formulation.

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7. **Dynamical symmetries: A purely algebraic solution for the hydrogen atom**.
Let us consider a particle of (reduced) mass m and position vector $\vec{r}(t) = \{x_i(t);\ i = 1, 2, 3\}$ in three euclidean space subjected to the Coulomb-Kepler central potential, with Lagrange function

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(r), \qquad V(r) = -\frac{\lambda}{r}, \qquad r = |\vec{r}\,|,$$

λ being a constant setting the strength of the central force. For positive (resp., negative) λ, this force is attractive (resp., repulsive). In the case of the Kepler problem for two massive bodies of masses m_1 and m_2 one has $\lambda = G_N m_1 m_2$ (in which case $m = m_1 m_2/(m_1 + m_2)$), while for the Coulomb problem for two electric charges Q_1 and Q_2 (in S.I. units) one has $\lambda = -Q_1 Q_2/(4\pi\epsilon_0)$. Note that for a hydrogenoid atom composed of a single electron of charge $-|e|$ and a point nucleus of total charge $Z|e| > 0$, one has

$$\lambda = \frac{Ze^2}{4\pi\epsilon_0} = \hbar c\, Z\alpha, \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137},$$

α being the atomic fine structure constant.

The Hamiltonian formulation of this dynamics is thus specified by the Hamiltonian

$$H = \frac{1}{2}\vec{p}^{\,2} + V(r) = \frac{1}{2m}\vec{p}^{\,2} - \frac{\lambda}{r},$$

with the canonical brackets for the conjugate variables (x_i, p_i), $\{x_i, p_j\} = \delta_{ij}$ $(i, j = 1, 2, 3)$. It is well known that besides the total energy, H, and orbital angular-momentum, $\vec{L} = \vec{r} \times \vec{p}$, namely $L_i = \epsilon_{ijk} x_j p_k = -\epsilon_{ijk} p_j x_k$, which are conserved quantities for this system, for this particular choice of interaction potential there exists a second conserved vector quantity, namely the Laplace–Runge–Lenz vector,

$$\vec{A} = \vec{p} \times \vec{L} - \lambda m\hat{r}, \qquad A_i = \epsilon_{ijk} p_j L_k - \lambda m\frac{x_i}{r} = -\epsilon_{ijk} L_j p_k - \lambda m\frac{x_i}{r} = \left\{\frac{1}{2}L^2 - \lambda m r, p_i\right\},$$

where $L^2 \equiv \vec{L}^2 \equiv L_i^2$.

The purpose of the present analysis is to extend to the quantum dynamics the existence of these conserved quantities, and even manage to compute the spectrum of negative energy bound states in the attractive case using purely algebraic considerations. In particular the existence of the Laplace–Runge–Lenz vector is related to a dynamical symmetry in this system, which accounts for the degeneracies of the energy spectrum of the hydrogen atom, for instance. In three euclidean space it turns out the dynamical symmetry extends the SO(3) rotational symmetry to a SO(4) symmetry for the bound states. A similar result generalises to an euclidean space of any dimension d with then the rotational symmetry SO(d) extended into a dynamical SO($d+1$) symmetry for the bound states.

The quantised system is defined by the Heisenberg algebra commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij},$$

while the above classically conserved quantities have the following quantum operator counterparts,

$$\hat{H} = \frac{1}{2m}\hat{p}_i^2 - \frac{\lambda}{\hat{r}}, \qquad \hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k = -\epsilon_{ijk}\hat{p}_j\hat{x}_k, \qquad \hat{A}_i = \frac{i}{\hbar}\left[\hat{p}_i, \frac{1}{2}\hat{L}^2 - \lambda m\hat{r}\right],$$

where $\hat{r} = \sqrt{\hat{x}_i^2}$ and $\hat{L}^2 \equiv \hat{L}_i^2$. Note that these choices all define hermitian and self-adjoint operators.

a. Using $[\hat{p}_i, \hat{r}] = -i\hbar\hat{x}_i/\hat{r}$, establish the following alternative expressions

$$\begin{aligned} \hat{A}_i &= \frac{1}{2}\left(\epsilon_{ijk}\hat{p}_j\hat{L}_k - \epsilon_{ijk}\hat{L}_j\hat{p}_k\right) - \lambda m\frac{\hat{x}_i}{\hat{r}} \\ &= \epsilon_{ijk}\hat{p}_j\hat{L}_k - i\hbar\hat{p}_i - \lambda m\frac{\hat{x}_i}{\hat{r}} = -\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i - \lambda m\frac{\hat{x}_i}{\hat{r}}, \end{aligned} \tag{1}$$

as well as the useful identities,

$$\hat{x}_i\hat{L}_i = 0 = \hat{L}_i\hat{x}_i, \qquad \hat{p}_i\hat{L}_i = 0 = \hat{L}_i\hat{p}_i, \qquad \hat{A}_i\hat{L}_i = 0 = \hat{L}_i\hat{A}_i.$$

b. Considering first the angular-momentum operator, one has the following properties under infinitesimal spatial rotations,

$$\left[\hat{L}_i, \hat{x}_j\right] = i\hbar\epsilon_{ijk}\hat{x}_k = \left[\hat{x}_i, \hat{L}_j\right], \qquad \left[\hat{L}_i, \hat{p}_j\right] = i\hbar\epsilon_{ijk}\hat{p}_k = \left[\hat{p}_i, \hat{L}_j\right],$$

hence,

$$\left[\hat{L}_i, \hat{x}_j^2\right] = 0 = \left[\hat{L}_i, \hat{p}_j^2\right],$$

as well as,

$$\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\epsilon_{ijk}\hat{L}_k.$$

As a consequence, show that $\hat{H}$ and $\hat{L}_i$ are indeed conserved quantities, namely

$$\left[\hat{H}, \hat{H}\right] = 0 = \left[\hat{L}_i, \hat{H}\right].$$

c. Turning now to the Laplace–Runge–Lenz vector, using one of its expanded expressions above (not in terms of the commutator definition), show that this quantum operator is also conserved,

$$\left[\hat{A}_i, \hat{H}\right] = 0.$$

Exploiting the expression of $\hat{A}_i$ in terms of a commutator and applying the Jacobi identity, establish that under spatial rotations these operators indeed define the components of a vector quantity,

$$\left[\hat{L}_i, \hat{A}_j\right] = i\hbar\epsilon_{ijk}\hat{A}_k = \left[\hat{A}_i, \hat{L}_j\right].$$

d. Using one of the expressions in (1) above, as well as $[\hat{p}_i, 1/\hat{r}] = i\hbar\hat{x}_i/\hat{r}^3$, through a careful and patient calculation, check that the Laplace–Runge–Lenz operator obeys the algebra,

$$\left[\hat{A}_i, \hat{A}_j\right] = -i\hbar\epsilon_{ijk}\left(2m\hat{H}\right)\hat{L}_k.$$

It thus proves useful to introduce the following normalised operators,

$$\hat{D}_i = \frac{1}{\sqrt{2m|\hat{H}|}}\hat{A}_i,$$

which are well defined since $\hat{H}$ commutes with both $\hat{L}_i$ and $\hat{A}_i$, and which are such that

$$\left[\hat{D}_i, \hat{D}_j\right] = -i\hbar\, s\, \epsilon_{ijk}\hat{L}_k, \qquad s = \mathrm{sgn}\,\hat{H}.$$

The operators $\hat{L}_i$ and $\hat{D}_i$ thus form an algebra which closes onto itself. For $s = -1$, namely bound states, this algebra is that of SU(2)×SU(2), namely SO(4), as will now be established. For $s = +1$, namely unbound states, the algebra is that of the noncompact group SO(1,3), which shall not be considered here.

e. When $s = -1$, consider the collection of operators $\hat{L}_{\alpha\beta}$ with $\alpha, \beta = 1, 2, 3, 4$ such that $\hat{L}_{\beta\alpha} = -\hat{L}_{\alpha\beta}$ and defined by

$$\hat{L}_{ij} = \epsilon_{ijk}\hat{L}_k, \qquad \hat{L}_{4i} = \hat{D}_i, \qquad i, j, k = 1, 2, 3.$$

Check that their algebra is

$$\left[\hat{L}_{\alpha\beta}, \hat{L}_{\gamma\delta}\right] = i\hbar\left(\delta_{\alpha\gamma}\hat{L}_{\beta\delta} - \delta_{\alpha\delta}\hat{L}_{\beta\gamma} - \delta_{\beta\gamma}\hat{L}_{\alpha\delta} + \delta_{\beta\delta}\hat{L}_{\alpha\gamma}\right),$$

which is in fact the SO(4) algebra in four dimension euclidean space. As a matter of fact the SO(4) algebra is isomorphic to the SU(2)× SU(2) algebra, as may be seen as follows. Defining the operators

$$\hat{R}_i = \frac{1}{2}\left(\hat{L}_i + \hat{D}_i\right), \qquad \hat{S}_i = \frac{1}{2}\left(\hat{L}_i - \hat{D}_i\right).$$

one easily finds these span two commuting SU(2) algebras,

$$\left[\hat{R}_i, \hat{R}_j\right] = i\hbar\epsilon_{ijk}\hat{R}_k, \qquad \left[\hat{S}_i, \hat{S}_j\right] = i\hbar\epsilon_{ijk}\hat{S}_k, \qquad \left[\hat{R}_i, \hat{S}_j\right] = 0,$$

each having its own quadratic Casimir operator, $\hat{R}^2 \equiv \hat{R}_i^2$ and $\hat{S}^2 \equiv \hat{S}_i^2$,

$$\left[\hat{R}_i, \hat{R}^2\right] = 0 = \left[\hat{S}_i, \hat{S}^2\right].$$

However, these two Casimir operators are not independent in the present system. Using the fact that $\hat{L}_i\hat{A}_i = 0 = \hat{A}_i\hat{L}_i$, show that

$$\hat{R}^2 - \hat{S}^2 = 0.$$

Consequently, bound quantum states may be characterised in terms of SU(2)×SU(2) quantum numbers related to representations of spin j_+, say for the $\hat{R}_i$ algebra, and of spin j_- for the $\hat{S}_i$ algebra, with however $j_+ = j_- = j$ and j being a positive integer or half-integer number.

f. Using the identity

$$\hat{A}_i + \lambda m\frac{\hat{x}_i}{\hat{r}} = \epsilon_{ijk}\hat{p}_j\hat{L}_k - i\hbar\hat{p}_i = -\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i,$$

from which follows the relation

$$\left(\hat{A}_i + \lambda m\frac{\hat{x}_i}{\hat{r}}\right)^2 = \left(\epsilon_{ijk}\hat{p}_j\hat{L}_k - i\hbar\hat{p}_i\right)\left(-\epsilon_{i\ell m}\hat{L}_\ell\hat{p}_m + i\hbar\hat{p}_i\right),$$

show that one has

$$\hat{A}_i^2 = 2m\hat{H}\left(\hat{L}^2 + \hbar^2\right) + (\lambda m)^2,$$

leading to,

$$\hat{H}=-\frac{1}{2}\frac{m\lambda^2}{\hbar^2}\,\frac{1}{1+\frac{1}{\hbar^2}\left(\hat{L}^2-s\hat{D}^2\right)},$$

with in particular for bound states, $s=-1$,

$$\hat{H}=-\frac{m\lambda^2}{2\hbar^2}\,\frac{1}{1+\frac{2}{\hbar^2}\left(\hat{R}^2+\hat{S}^2\right)}.$$

Conclude that the energy spectrum of the system is given by

$$E_j=-\frac{m\lambda^2}{2\hbar^2}\,\frac{1}{(2j+1)^2},\qquad j=0,\frac{1}{2},1,\frac{3}{2},2,\dots,$$

each such level having a degeneracy $(2j+1)^2$. This result indeed explains the degenerate spectrum of the hydrogen atom, with n^2 states at level $E_n=-\frac{1}{2}\left(Z\alpha\right)^2 mc^2\frac{1}{n^2}$ and $n=1,2,3,\dots$.

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a. Since the quantum coordinates $\hat{x}_i$ commute with one another, given any function $f(r)$ with $r=\sqrt{x_i^2}$, one has for the operator $f(\hat{r})$,

$$[\hat{p}_i,f\left(\hat{r}\right)]=\left[\hat{p}_i,\hat{r}^2\right]\,\frac{df(r)}{dr^2}_{|r=\hat{r}}=-2i\hbar\hat{x}_i\,\frac{1}{2\hat{r}}f'\left(\hat{r}\right)=-i\hbar\frac{\hat{x}_i}{\hat{r}}f'\left(\hat{r}\right). \tag{2}$$

Hence in particular,

$$[\hat{p}_i,\hat{r}]=-i\hbar\frac{\hat{x}_i}{\hat{r}}. \tag{3}$$

Let us now consider the expression for the quantum Laplace–Runge–Lenz (LRL) vector,

$$\hat{A}_i=\frac{i}{\hbar}\left[\hat{p}_i,\frac{1}{2}\hat{L}^2-\lambda m\hat{r}\right]. \tag{4}$$

Since $\hat{L}^2\equiv\hat{L}_i^2$, in order to evaluate this commutator, one needs first to consider the following ones,

$$\left[\hat{p}_i,\hat{L}_j\right]=\epsilon_{jk\ell}\left[\hat{p}_i,\hat{x}_k\hat{p}_\ell\right]=-i\hbar\epsilon_{jk\ell}\delta_{ik}\hat{p}_\ell=i\hbar\epsilon_{ijk}\hat{p}_k. \tag{5}$$

Hence we directly have,

$$\begin{aligned}\hat{A}_i&=\frac{i}{2\hbar}\left[\hat{p}_i,\hat{L}_j^2\right]-\frac{i}{\hbar}\lambda m\left[\hat{p}_i,\hat{r}\right]\\&=\frac{i}{2\hbar}\left(\left[\hat{p}_i,\hat{L}_j\right]\hat{L}_j+\hat{L}_j\left[\hat{p}_i,\hat{L}_j\right]\right)-\lambda m\frac{\hat{x}_i}{\hat{r}}\\&=\frac{1}{2}\left(\epsilon_{ijk}\hat{p}_j\hat{L}_k-\epsilon_{ijk}\hat{L}_j\hat{p}_k\right)-\lambda m\frac{\hat{x}_i}{\hat{r}},\end{aligned} \tag{6}$$

Furthermore one also finds,

$$\epsilon_{ijk}\hat{p}_j\hat{L}_k=\epsilon_{ijk}\left[\hat{p}_j,\hat{L}_k\right]+\epsilon_{ijk}\hat{L}_k\hat{p}_j=i\hbar\epsilon_{ijk}\epsilon_{jk\ell}\hat{p}_\ell-\epsilon_{ijk}\hat{L}_j\hat{p}_k=2i\hbar\hat{p}_i-\epsilon_{ijk}\hat{L}_j\hat{p}_k, \tag{7}$$

having used the following identity, $\epsilon_{ik\ell}\epsilon_{jk\ell}=2\delta_{ij}$. Conversely,

$$\epsilon_{ijk}\hat{L}_j\hat{p}_k=2i\hbar\hat{p}_i-\epsilon_{ijk}\hat{p}_j\hat{L}_k. \tag{8}$$

Consequently one may also write

$$\hat{A}_i = \epsilon_{ijk}\hat{p}_k\hat{L}_k \, - \, i\hbar\hat{p}_i \, - \, \lambda m\frac{\hat{x}_i}{\hat{r}} = -\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i - \lambda m\frac{\hat{x}_i}{\hat{r}}. \tag{9}$$

Let us now consider also the scalar products of the different vector quantities. One has

$$\hat{x}_i\hat{L}_i = \epsilon_{ijk}\hat{x}_i\hat{x}_j\hat{p}_k = 0, \qquad \hat{L}_i\hat{x}_i = -\epsilon_{ijk}\hat{p}_j\hat{x}_k\hat{x}_i = 0, \tag{10}$$

$$\hat{p}_i\hat{L}_i = -\epsilon_{ijk}\hat{p}_i\hat{p}_j\hat{x}_k = 0, \qquad \hat{L}_i\hat{p}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k\hat{p}_i = 0, \tag{11}$$

where in each case one obtains a vanishing quantity on account of the antisymmetry of ϵ_{ijk} while the products of two $\hat{x}_i$'s or two $\hat{p}_k$'s are symmetric in their two summed indices since the factors in these products commute with one another. For the same reason one also has

$$\hat{A}_i\hat{L}_i = \left(\epsilon_{ijk}\hat{p}_k\hat{L}_k \, - \, i\hbar\hat{p}_i \, - \, \lambda m\frac{\hat{x}_i}{\hat{r}}\right)\hat{L}_i = 0, \qquad \hat{L}_i\hat{A}_i = \hat{L}_i\left(-\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i - \lambda m\frac{\hat{x}_i}{\hat{r}}\right) = 0. \tag{12}$$

These identities are useful in later calculations.

b. It has already been established above that one has

$$\left[\hat{p}_i, \hat{L}_j\right] = i\hbar\epsilon_{ijk}\hat{p}_k. \tag{13}$$

Consequently,

$$\left[\hat{L}_i, \hat{p}_j\right] = -\left[\hat{p}_j, \hat{L}_i\right] = -i\hbar\epsilon_{jik}\hat{p}_k = i\hbar\epsilon_{ijk}\hat{p}_k. \tag{14}$$

Likewise,

$$\left[\hat{L}_i, \hat{x}_j\right] = \epsilon_{ik\ell}\left[\hat{x}_k\hat{p}_\ell, \hat{x}_j\right] = -i\hbar\epsilon_{ik\ell}\delta_{\ell j}\hat{x}_k = i\hbar\epsilon_{ijk}\hat{x}_k, \tag{15}$$

and thus,

$$\left[\hat{x}_i, \hat{L}_j\right] = -\left[\hat{L}_j, \hat{x}_i\right] = -i\hbar\epsilon_{jik}\hat{x}_k = i\hbar\epsilon_{ijk}\hat{x}_k. \tag{16}$$

The meaning of these commutators is that under rotations generated by the angular-momentum operator, the operators $\hat{x}_i$ and $\hat{p}_i$ transform like vector quantities. Consequently, the scalar products of such vector quantities should also be invariants under the rotations, namely commute with the angular-momentum operators. Indeed, one has,

$$\left[\hat{L}_i, \hat{x}_j^2\right] = \left[\hat{L}_i, \hat{x}_j\right]\hat{x}_j + \hat{x}_j\left[\hat{L}_i, \hat{x}_j\right] = 2i\hbar\epsilon_{ijk}\hat{x}_j\hat{x}_k = 0, \tag{17}$$

$$\left[\hat{L}_i, \hat{p}_j^2\right] = \left[\hat{L}_i, \hat{p}_j\right]\hat{p}_j + \hat{p}_j\left[\hat{L}_i, \hat{p}_j\right] = 2i\hbar\epsilon_{ijk}\hat{p}_j\hat{p}_k = 0, \tag{18}$$

where in each case one obtains a vanishing quantity on account of the antisymmetry of ϵ_{ijk} in the two indices j and k while the products $\hat{x}_j\hat{x}_k$ and $\hat{p}_j\hat{p}_k$ are symmetric in these two summed indices. Likewise, it follows that,

$$\left[\hat{L}_i, \hat{x}_j\hat{p}_j\right] = i\hbar\epsilon_{ijk}\hat{x}_k\hat{p}_j + i\hbar\epsilon_{ijk}\hat{x}_j\hat{p}_k = 0, \tag{19}$$

$$\left[\hat{L}_i, \hat{p}_j\hat{x}_j\right] = i\hbar\epsilon_{ijk}\hat{p}_k\hat{x}_j + i\hbar\epsilon_{ijk}\hat{p}_j\hat{x}_k = 0. \tag{20}$$

Using the identity

$$\epsilon_{ik\ell}\epsilon_{jm\ell} = \delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}, \tag{21}$$

one then also finds,

$$\begin{aligned}\left[\hat{L}_i,\hat{L}_j\right] &= \epsilon_{jk\ell}\left[\hat{L}_i,\hat{x}_k\hat{p}_\ell\right] = i\hbar\epsilon_{jk\ell}\epsilon_{ikm}\hat{x}_m\hat{p}_\ell + i\hbar\epsilon_{jk\ell}\epsilon_{i\ell m}\hat{x}_k\hat{p}_m \\ &= i\hbar\left(\delta_{ji}\delta_{\ell m} - \delta_{jm}\delta_{\ell i}\right)\hat{x}_m\hat{p}_\ell + i\hbar\left(-\delta_{ji}\delta_{km} + \delta_{jm}\delta_{ki}\right)\hat{x}_k\hat{p}_m \\ &= i\hbar\left(\delta_{ij}\hat{x}_k\hat{p}_k - \hat{x}_j\hat{p}_i - \delta_{ij}\hat{x}_k\hat{p}_k + \hat{x}_i\hat{p}_j\right) \\ &= i\hbar\epsilon_{ijk}\hat{L}_k, \end{aligned} \tag{22}$$

showing that the components of the angular-momentum operator also transform as a vector under rotations, the operators $\hat{L}_i$ thus spanning the SO(3) algebra of three dimensional euclidean space. In particular, once again this implies that $\hat{L}^2 \equiv \hat{L}_i^2$ is invariant under such rotations,

$$\left[\hat{L}_i,\hat{L}^2\right] = 0. \tag{23}$$

From all these different results it readily follows that

$$\left[\hat{H},\hat{H}\right] = 0, \qquad \left[\hat{L}_i,\hat{H}\right] = 0, \tag{24}$$

thus expressing the fact that $\hat{H}$ and $\hat{L}_i$ are indeed conserved quantities, with $\hat{H}$ being invariant under SO(3) rotations.

c. In order to establish the conservation of the LRL vector $\hat{A}_i$, let us consider its expression in (6). For each of the three contributions to that expression, we have,

$$\begin{aligned}\left[\epsilon_{ijk}\hat{p}_j\hat{L}_k,\hat{H}\right] &= -\lambda\epsilon_{ijk}\left[\hat{p}_j\hat{L}_k,\frac{1}{\hat{r}}\right] = -i\hbar\lambda\epsilon_{ijk}\frac{1}{\hat{r}^3}\hat{x}_j\hat{L}_k \\ &= -i\hbar\lambda\epsilon_{ijk}\frac{1}{\hat{r}^3}\hat{x}_j\epsilon_{k\ell m}\hat{x}_\ell\hat{p}_m \\ &= -i\hbar\lambda\frac{1}{\hat{r}^3}\left(\hat{x}_j\hat{x}_i\hat{p}_j - \hat{x}_j^2\hat{p}_i\right) = -i\hbar\lambda\left(\frac{1}{\hat{r}^3}\hat{x}_i\hat{x}_j\hat{p}_j - \frac{1}{\hat{r}}\hat{p}_i\right), \end{aligned} \tag{25}$$

$$\begin{aligned}\left[\epsilon_{ijk}\hat{L}_j\hat{p}_k,\hat{H}\right] &= -\lambda\epsilon_{ijk}\left[\hat{L}_j\hat{p}_k,\frac{1}{\hat{r}}\right] = -i\hbar\lambda\epsilon_{ijk}\hat{L}_j\hat{x}_k\frac{1}{\hat{r}^3} \\ &= i\hbar\lambda\epsilon_{ijk}\epsilon_{j\ell m}\hat{p}_\ell\hat{x}_m\hat{x}_k\frac{1}{\hat{r}^3} \\ &= i\hbar\lambda\left(-\hat{p}_i\hat{x}_j^2 + \hat{p}_j\hat{x}_i\hat{x}_j\right) = i\hbar\lambda\left(\hat{p}_j\hat{x}_j\hat{x}_i\frac{1}{\hat{r}^3} - \hat{p}_i\frac{1}{\hat{r}}\right), \end{aligned} \tag{26}$$

$$\begin{aligned}\left[\frac{\hat{x}_i}{\hat{r}},\hat{H}\right] &= \frac{1}{2m}\left[\frac{\hat{x}_i}{\hat{r}},\hat{p}_j^2\right] = \frac{1}{2m}\left[\frac{\hat{x}_i}{\hat{r}},\hat{p}_j\right]\hat{p}_j + \frac{1}{2m}\hat{p}_j\left[\frac{\hat{x}_i}{\hat{r}},\hat{p}_j\right] \\ &= \frac{1}{2m}i\hbar\left(\frac{1}{\hat{r}}\hat{p}_i - \frac{1}{\hat{r}^3}\hat{x}_i\hat{x}_j\hat{p}_j\right) + \frac{1}{2m}i\hbar\left(\hat{p}_i\frac{1}{\hat{r}} - \hat{p}_j\hat{x}_j\hat{x}_i\frac{1}{\hat{r}^3}\right). \end{aligned} \tag{27}$$

Consequently it readily follows that, by combining all contributions to $\hat{A}_i$,

$$\left[\hat{A}_i,\hat{H}\right] = 0. \tag{28}$$

Hence even at the quantum level the Laplace–Runge–Lenz vector as an operator is a conserved quantity.

In order to commute the transformation properties under spatial rotations of the LRL vector, it proves useful to consider its expression in terms of the commutator in (4), since the Jacobi identity then readily leads to the final expression,

$$\begin{aligned}\left[\hat{L}_i,\hat{A}_j\right] &= \frac{i}{\hbar}\left[\hat{L}_i,\left[\hat{p}_j,\frac{1}{2}\hat{L}^2-\lambda m\hat{r}\right]\right]\\ &= -\frac{i}{\hbar}\left[\hat{p}_j,\left[\frac{1}{2}\hat{L}^2-\lambda m\hat{r},\hat{L}_i\right]\right]-\frac{i}{\hbar}\left[\frac{1}{2}\hat{L}^2-\lambda m\hat{r},\left[\hat{L}_i,\hat{p}_j\right]\right]\\ &= -\frac{i}{\hbar}i\hbar\epsilon_{ijk}\left[\frac{1}{2}\hat{L}^2-\lambda m\hat{r},\hat{p}_k\right]\\ &= i\hbar\epsilon_{ijk}\hat{A}_k.\end{aligned}\tag{29}$$

In conclusion,

$$\left[\hat{L}_i,\hat{A}_j\right]=i\hbar\epsilon_{ijk}\hat{A}_k,\qquad \left[\hat{A}_i,\hat{L}_j\right]=i\hbar\epsilon_{ijk}\hat{A}_k.\tag{30}$$

d. The evaluation of the commutator $\left[\hat{A}_i,\hat{A}_j\right]$ proves to be the most technical of the entire analysis. Let us consider the expression for the LRL vector in the form

$$\hat{A}_i=\epsilon_{ijk}\hat{p}_j\hat{L}_k-i\hbar\hat{p}_i-\lambda m\frac{\hat{x}_i}{\hat{r}}.\tag{31}$$

Through a careful and detailed calculation one finds,

$$\left[\epsilon_{ik\ell}\hat{p}_k\hat{L}_\ell,\epsilon_{jmn}\hat{p}_m\hat{L}_n\right]=-i\hbar\epsilon_{ijk}\hat{p}_\ell^2\hat{L}_k,\tag{32}$$

as well as,

$$\left[\epsilon_{ik\ell}\hat{p}_k\hat{L}_\ell,-i\hbar\hat{p}_j\right]=\hbar^2\left(\delta_{ij}\hat{p}_\ell^2-\hat{p}_i\hat{p}_j\right),\tag{33}$$

$$\left[-i\hbar\hat{p}_i,\epsilon_{jk\ell}\hat{p}_k\hat{L}_\ell\right]=-\hbar^2\left(\delta_{ij}\hat{p}_\ell^2-\hat{p}_i\hat{p}_j\right).\tag{34}$$

Consequently,

$$\left[\epsilon_{ik\ell}\hat{p}_k\hat{L}_\ell-i\hbar\hat{p}_i,\epsilon_{jmn}\hat{p}_m\hat{L}_n-i\hbar\hat{p}_j\right]=-i\hbar\epsilon_{ijk}\hat{p}_\ell^2\hat{L}_k,\tag{35}$$

leading to the intermediate result,

$$\left[\hat{A}_i,\hat{A}_j\right]=-i\hbar\epsilon_{ijk}\hat{p}_\ell^2\hat{L}_k-\lambda m\left\{\left[\epsilon_{ik\ell}\hat{p}_k\hat{L}_\ell-i\hbar\hat{p}_i,\frac{\hat{x}_j}{\hat{r}}\right]-(i\leftrightarrow j)\right\}.\tag{36}$$

Since one has

$$\left[-i\hbar\hat{p}_i,\frac{\hat{x}_j}{\hat{r}}\right]=\hbar^2\left(\frac{\hat{x}_i\hat{x}_j}{\hat{r}^3}-\delta_{ij}\frac{1}{\hat{r}}\right),\tag{37}$$

it follows that

$$\left[-i\hbar\hat{p}_i,\frac{\hat{x}_j}{\hat{r}}\right]-(i\leftrightarrow j)=0.\tag{38}$$

Furthermore a detailed calculation finds,

$$\left[\epsilon_{ik\ell}\hat{p}_k\hat{L}_\ell,\frac{\hat{x}_j}{\hat{r}}\right]=-i\hbar\epsilon_{ijk}\frac{1}{\hat{r}}\hat{L}_k+i\hbar\left(\delta_{ij}\hat{p}_k\hat{x}_k\frac{1}{\hat{r}}-\hat{p}_j\hat{x}_i\frac{1}{\hat{r}}+\frac{1}{\hat{r}^3}\epsilon_{ik\ell}\hat{x}_j\hat{x}_k\hat{L}_\ell\right),\tag{39}$$

so that

$$\left[\epsilon_{ik\ell}\hat{p}_k\hat{L}_\ell,\frac{\hat{x}_j}{\hat{r}}\right]-(i\leftrightarrow j)=-2i\hbar\epsilon_{ijk}\frac{1}{\hat{r}}\hat{L}_k+i\hbar\left[\left(\hat{p}_i\hat{x}_j-\hat{p}_j\hat{x}_i\right)\frac{1}{\hat{r}}+\frac{1}{\hat{r}^3}\left(\epsilon_{ik\ell}\hat{x}_j\hat{x}_k\hat{L}_\ell-\epsilon_{jk\ell}\hat{x}_i\hat{x}_k\hat{L}_\ell\right)\right].\tag{40}$$

An explicit evaluation of the latter two terms leads to,

$$\epsilon_{ik\ell}\hat{x}_j\hat{x}_k\hat{L}_\ell = \hat{x}_i\hat{x}_j\hat{x}_k\hat{p}_k - \hat{r}^2\hat{x}_j\hat{p}_j, \tag{41}$$

$$\epsilon_{ik\ell}\hat{x}_j\hat{x}_k\hat{L}_\ell - \epsilon_{jk\ell}\hat{x}_i\hat{x}_k\hat{L}_\ell = \hat{r}^2\left(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i\right). \tag{42}$$

It then follows that the last term in square brackets in (40) reduces to,

$$\begin{aligned} i\hbar\left[(\hat{p}_i\hat{x}_j - \hat{p}_j\hat{x}_i)\frac{1}{\hat{r}} + \frac{1}{\hat{r}}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)\right] &= i\hbar\left(-i\hbar\delta_{ij}\frac{1}{\hat{r}} + i\hbar\delta_{ij}\frac{1}{\hat{r}} + \hat{x}_j\left[\hat{p}_i, \frac{1}{\hat{r}}\right] - \hat{x}_i\left[\hat{p}_j, \frac{1}{\hat{r}}\right]\right) \\ &= i\hbar\left(\hat{x}_j\, i\hbar\frac{\hat{x}_i}{\hat{r}^3} - \hat{x}_i\, i\hbar\frac{\hat{x}_j}{\hat{r}^3}\right) = 0. \end{aligned} \tag{43}$$

Hence finally we have obtained,

$$\left[\hat{A}_i, \hat{A}_j\right] = -i\hbar\epsilon_{ijk}\hat{p}_\ell^2\hat{L}_k - i\hbar\epsilon_{ijk}\left(-2\lambda m\right)\frac{1}{\hat{r}}\hat{L}_k = -i\hbar\epsilon_{ijk}\left(2m\hat{H}\right)\hat{L}_k. \tag{44}$$

Since $\hat{H}$ commutes with all the operators $\hat{A}_i$ and $\hat{L}_i$, it is possible to define

$$\hat{D}_i = \frac{1}{\sqrt{2m|\hat{H}|}}\hat{A}_i, \qquad s = \mathrm{sgn}\,\hat{H}, \tag{45}$$

so that the final algebra of interest for the LRL vector is

$$\left[\hat{D}_i, \hat{D}_j\right] = -i\hbar s\epsilon_{ijk}\hat{L}_k. \tag{46}$$

Together with the other commutation relations,

$$\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\epsilon_{ijk}\hat{L}_k, \qquad \left[\hat{L}_i, \hat{D}_j\right] = i\hbar\epsilon_{ijk}\hat{D}_k, \qquad \left[\hat{D}_i, \hat{L}_j\right] = i\hbar\epsilon_{ijk}\hat{D}_k, \tag{47}$$

the set of operators $\hat{L}_i$ and $\hat{D}_i$ thus forms a closed algebra for any subspace of Hilbert space associated to a given energy eigenvalue. This is the algebra of a symmetry which accounts for the degeneracy of energy eigenstates for the system.

e. Let us consider specifically those states for which $s = -1$, namely the bound states of the system, which requires that $\lambda > 0$. Introduce then the notations,

$$\hat{L}_{ij} = \epsilon_{ijk}\hat{L}_k, \qquad \hat{L}_{4i} = \hat{D}_i, \qquad \hat{L}_{i4} = -\hat{D}_i, \tag{48}$$

the ensemble of these operators being then denoted

$$\hat{L}_{\alpha\beta} = -\hat{L}_{\beta\alpha}, \qquad \alpha, \beta = 1, 2, 3, 4. \tag{49}$$

That these operators span the following SO(4) algebra for the generators of rotations in a four dimensional euclidean space,

$$\left[\hat{L}_{\alpha\beta}, \hat{L}_{\gamma\delta}\right] = i\hbar\left(\delta_{\alpha\gamma}\hat{L}_{\beta\delta} - \delta_{\alpha\delta}\hat{L}_{\beta\gamma} - \delta_{\beta\gamma}\hat{L}_{\alpha\delta} + \delta_{\beta\delta}\hat{L}_{\alpha\gamma}\right), \tag{50}$$

may be checked as follows. Given these last relations, one has,

$$\left[\hat{L}_{ij}, \hat{L}_{k\ell}\right] = i\hbar\left(\delta_{ik}\hat{L}_{j\ell} - \delta_{i\ell}\hat{L}_{jk} - \delta_{jk}\hat{L}_{i\ell} + \delta_{j\ell}\hat{L}_{ik}\right), \tag{51}$$

so that, after a little algebra, and using $\hat{L}_i = \frac{1}{2}\epsilon_{ijk}\hat{L}_{jk}$,

$$\left[\hat{L}_m, \hat{L}_n\right] = \frac{1}{4}\epsilon_{mij}\epsilon_{nk\ell}\left[\hat{L}_{ij}, \hat{L}_{k\ell}\right] = i\hbar\epsilon_{mnr}\hat{L}_r, \tag{52}$$

which is indeed the correct SO(3) angular-momentum algebra. Likewise,

$$\left[\hat{L}_{ij}, \hat{L}_{4k}\right] = i\hbar\left(-\delta_{ik}\hat{L}_{j4} + \delta_{jk}\hat{L}_{i4}\right) = i\hbar\left(\delta_{ik}\hat{D}_i - \delta_{jk}\hat{D}_i\right), \tag{53}$$

so that,

$$\left[\hat{L}_\ell, \hat{D}_k\right] = \frac{1}{2}\epsilon_{\ell ij}\left[\hat{L}_{ij}, \hat{L}_{4k}\right] = i\hbar\epsilon_{\ell km}\hat{D}_m, \tag{54}$$

which is indeed the correct commutator. And finally,

$$\left[\hat{D}_i, \hat{D}_j\right] = \left[\hat{L}_{4i}, \hat{L}_{4j}\right] = i\hbar\hat{L}_{ij} = i\hbar\epsilon_{ijk}\hat{L}_k, \tag{55}$$

as it should.

Hence indeed for any given bound state energy level, the six independent generators $\hat{L}_i$ and $\hat{D}_i$, namely $\hat{L}_{\alpha\beta}$, span the SO(4) algebra in (50). That this algebra is also that of the product SU(2)×SU(2) of two SU(2) algebras is seen as follows. Introduce

$$\hat{R}_i = \frac{1}{2}\left(\hat{L}_i + \hat{D}_i\right), \qquad \hat{S}_i = \frac{1}{2}\left(\hat{L}_i - \hat{D}_i\right), \tag{56}$$

$$\hat{L}_i = \hat{R}_i + \hat{S}_i, \qquad \hat{D}_i = \hat{R}_i - \hat{S}_i. \tag{57}$$

A direct calculation the gives,

$$\left[\hat{R}_i, \hat{R}_j\right] = i\hbar\epsilon_{ijk}\hat{R}_k, \qquad \left[\hat{R}_i, \hat{S}_j\right] = 0, \qquad \left[\hat{S}_i, \hat{S}_j\right] = i\hbar\epsilon_{ijk}\hat{S}_k. \tag{58}$$

Hence indeed each of the three sets of operators $\hat{R}_i$ and $\hat{S}_i$ define a SU(2) algebra, and these two algebras commute with one another. Note that since $\hat{A}_i\hat{L}_i = 0 = \hat{L}_i\hat{A}_i$, we also have $\hat{D}_i\hat{L}_i = 0 = \hat{L}_i\hat{D}_i$, hence

$$0 = \hat{D}_i\hat{L}_i = \left(\hat{R}_i - \hat{S}_i\right)\left(\hat{R}_i + \hat{S}_i\right) = \hat{R}^2 - \hat{S}^2, \qquad 0 = \hat{L}_i\hat{D}_i = \left(\hat{R}_i + \hat{S}_i\right)\left(\hat{R}_i - \hat{S}_i\right) = \hat{R}^2 - \hat{S}^2, \tag{59}$$

where $\hat{R}^2 \equiv \hat{R}_i^2$ and $\hat{S}^2 \equiv \hat{S}_i^2$ define the Casimir operators of the two SU(2) algebras,

$$\left[\hat{R}_i, \hat{R}^2\right] = 0 = \left[\hat{S}_i, \hat{S}^2\right]. \tag{60}$$

Since SU(2) representations are characterised by their Casimir eigenvalues,

$$\hat{R}^2 \;:\quad j_+(j_+ + 1)\hbar^2; \qquad \hat{S}^2 \;:\quad j_-(j_- + 1)\hbar^2, \tag{61}$$

j_+ and j_- being positive integers or half-integers, in the present system each bound state energy level is characterised by a common SU(2) spin value taking an integer or half-integer value, namely $j = j_+ = j_-$, corresponding to the SU(2)×SU(2) representation $(j_+, j_-) = (j, j)$. Hence each bound state energy level corresponds to a (j, j) representation of the SU(2)×SU(2)=SO(4) dynamical symmetry of the system. Since a SU(2) representation of spin j has dimension $(2j+1)$, each such bound state energy level has a degeneracy $(2j+1)^2$.

f. The only thing left doing now is identify the relation of the energy spectrum values to the generators of the dynamical SO(4) symmetry. For this purpose it is useful to consider the calculation of $\hat{A}_i^2$ in the following way. Given the two representations in (9) for the LRL vector, one has

$$\hat{A}_i + \lambda m\frac{\hat{x}_i}{\hat{r}} = \epsilon_{ijk}\hat{p}_k\hat{L}_k - i\hbar\hat{p}_i = -\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i, \tag{62}$$

hence,

$$\left(\hat{A}_i + \lambda m\frac{\hat{x}_i}{\hat{r}}\right)^2 = \left(\epsilon_{ijk}\hat{p}_k\hat{L}_k - i\hbar\hat{p}_i\right)\left(-\epsilon_{ijk}\hat{L}_j\hat{p}_k + i\hbar\hat{p}_i\right). \tag{63}$$

A straightforward calculation finds that,

$$\left(\epsilon_{ijk}\hat{p}_k\hat{L}_k\ -\ i\hbar\hat{p}_i\right)\left(-\epsilon_{ijk}\hat{L}_j\hat{p}_k+i\hbar\hat{p}_i\right)=\hat{p}_i^2\left(\hat{L}^2+\hbar^2\right). \tag{64}$$

On the other hand,

$$\left(\hat{A}_i\ +\ \lambda m\frac{\hat{x}_i}{\hat{r}}\right)^2=\hat{A}_i^2+\lambda m\left(\frac{1}{\hat{r}}\hat{x}_i\hat{A}_i+\hat{A}_i\hat{x}_i\frac{1}{\hat{r}}\right)+(\lambda m)^2, \tag{65}$$

of which the explicit evaluation gives,

$$\left(\hat{A}_i\ +\ \lambda m\frac{\hat{x}_i}{\hat{r}}\right)^2=\hat{A}_i^2+\lambda m\frac{1}{\hat{r}}\left(\hat{r}^2\hat{p}_i^2-(\hat{x}_i\hat{p}_i)(\hat{x}_j\hat{p}_j)\right)+\lambda m\left(\hat{p}_i^2\hat{r}^2-(\hat{p}_i\hat{x}_i)(\hat{p}_j\hat{x}_j)\right)\frac{1}{\hat{r}}-(\lambda m)^2. \tag{66}$$

However since,

$$\hat{L}^2=\hat{x}_i^2\hat{p}_j^2-(\hat{x}_i\hat{p}_i)(\hat{x}_j\hat{p}_j)+i\hbar\hat{x}_i\hat{p}_i=\hat{r}^2\hat{p}_i^2-(\hat{x}_i\hat{p}_i)(\hat{x}_j\hat{p}_j), \tag{67}$$

$$\hat{L}^2=\hat{p}_i^2\hat{x}_j^2-(\hat{p}_i\hat{x}_i)(\hat{p}_j\hat{x}_j)-i\hbar\hat{p}_i\hat{x}_i=\hat{p}_i^2\hat{r}^2-(\hat{p}_i\hat{x}_i)(\hat{p}_j\hat{x}_j)-i\hbar(\hat{p}_i\hat{x}_i), \tag{68}$$

one has so far,

$$\hat{A}_i^2=\hat{p}_i^2\left(\hat{L}^2+\hbar^2\right)-\lambda m\frac{1}{\hat{r}}\left(\hat{L}^2-i\hbar\hat{x}_i\hat{p}_i\right)-\lambda m\left(\hat{L}^2+i\hbar\hat{p}_i\hat{x}_i\right)\frac{1}{\hat{r}}+(\lambda m)^2. \tag{69}$$

Finally using the fact that

$$\frac{1}{\hat{r}}\hat{x}_i\hat{p}_i=3i\hbar\frac{1}{\hat{r}}+\frac{1}{\hat{r}}\hat{p}_i\hat{x}_i=3i\hbar\frac{1}{\hat{r}}-i\hbar\hat{x}_i\frac{1}{\hat{r}^3}\hat{x}_i+\hat{p}_i\frac{\hat{x}_i}{\hat{r}}=\hat{p}_i\hat{x}_i\frac{1}{\hat{r}}+2i\hbar\frac{1}{\hat{r}}, \tag{70}$$

it follows that,

$$\hat{A}^2\equiv\hat{A}_i^2=2m\left(\frac{1}{2m}\hat{p}_i^2-\frac{\lambda}{\hat{r}}\right)\left(\hat{L}^2+\hbar^2\right)+(\lambda m)^2=2m\hat{H}\left(\hat{L}^2+\hbar^2\right)+(\lambda m)^2. \tag{71}$$

Since $\hat{A}^2=2m|\hat{H}|\hat{D}^2$ with $\hat{D}^2\equiv\hat{D}_i^2$, the last result may be inverted to give the energy spectrum in the form,

$$\hat{H}=-\frac{1}{2}\frac{m\lambda^2}{\hbar^2}\frac{1}{1+\frac{1}{\hbar^2}\left(\hat{L}^2-s\hat{D}^2\right)}. \tag{72}$$

Focusing now on bound states only, $s=-1$, and noting that under this condition one has

$$\hat{L}^2-s\hat{D}^2=\hat{L}^2+\hat{D}^2=\left(\hat{R}_i+\hat{S}_i\right)^2+\left(\hat{R}_i-\hat{S}_i\right)^2=2\left(\hat{R}^2+\hat{S}^2\right), \tag{73}$$

which is thus related to the two Casimir invariants of the two SU(2) factors of the dynamical symmetry, one has for the bound state energy spectrum,

$$\hat{H}=-\frac{m\lambda^2}{2\hbar^2}\frac{1}{1+\frac{2}{\hbar^2}\left(\hat{R}^2+\hat{S}^2\right)}. \tag{74}$$

Consequently any eigen-energy sector related to the SU(2)×SU(2) representation $(j_+,j_-)=(j,j)$ has the following energy eigenvalue,

$$E(j)=-\frac{m\lambda^2}{2\hbar^2}\frac{1}{1+4j(j+1)}=-\frac{m\lambda^2}{2\hbar^2}\frac{1}{(2j+1)^2}, \tag{75}$$

where $(2j+1)^2$ is also the degeneracy of that bound state energy level. Since j takes all positive integer and half-integer values, $(2j+1)$ takes all strictly positive integer values, $n = 2j+1 = 1, 2, 3, \ldots$ In conclusion, the bound state energy spectrum has values

$$E_n = -\frac{1}{2}\frac{m\lambda^2}{\hbar^2}\frac{1}{n^2}, \qquad n = 1, 2, 3, \ldots, \tag{76}$$

each energy level having a degeneracy given by n^2, which corresponds to the dimension of the representation of the SO(4)=SU(2)×SU(2) dynamical symmetry of the system associated to that energy level.

Note that in the case of hydrogenoid atoms, this spectrum reads,

$$E_n = -\frac{1}{2}mc^2\,(Z\alpha)^2\,\frac{1}{n^2}, \tag{77}$$

which is indeed the expression for the energy values of the bound states of the hydrogen atom (then with $Z = 1$), with the correct degeneracies of states, which in atomic physics need to be lifted through spin-orbit interactions in order to explain Mendeleev's periodic table for the chemical elements.

B. Special Relativity

1. In its proper inertial frame R' a stick of length L_0 is positioned in the plane $(x'y')$ with an angle θ_0 with respect to the axis $x' > 0$. The frame R' is itself in uniform rectilinear motion with velocity $v_0 > 0$ with respect to the axis $x > 0$ of an inertial frame R of which all coordinates axes (xyz) coincide with those $(x'y'z')$ of the frame R'.

   a. In the frame R what is the value of the angle θ of the stick with the axis $x > 0$? (One sets of course $\gamma_0 = 1/\sqrt{1-\beta_0^2}$ with $\beta_0 = v_0/c$).

   b. What is the stick's length measured in the frame?

---

a. In the inertial frame R' the components of the stick that are parallel and transverse to the x' direction are, respectively,

$$L_0 \cos\theta_0, \qquad L_0 \sin\theta_0.$$

Likewise in the inertial frame R and with respect to the same x direction, these components are, respectively,

$$L\cos\theta, \qquad L\sin\theta.$$

However it is known that because of the stick's motion relative to R, the parallel component is contracted by the factor $1/\gamma_0$, while the transverse component remains unaffected. Hence one has

$$L\cos\theta = \frac{1}{\gamma_0} L_0 \cos\theta_0, \qquad L\sin\theta = L_0 \sin\theta_0. \tag{1}$$

Taking the ratio of the second with the first of this set of relations, it readily follows that,

$$\tan\theta = \gamma_0 \tan\theta_0.$$

In other words, seen from R, it is as if the stick would "stand up" in the direction transverse to the motion, since only its component parallel to that motion gets contracted.

b. Likewise taking the sum of squares of the two relations in (1), it readily follows that

$$L = L_0\sqrt{1-\beta_0^2\cos^2\theta_0}.$$

Note well that this is the only expression which may be consistent with a proper length contraction of the stick by a factor $\sqrt{1-\beta_0^2}$ when it moves parallel to itself ($\theta_0 = 0$), while its length remains unaffected when it moves perpendicularly to itself ($\theta_0 = \pi/2$ radians).

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2. **Composition of Lorentz boosts and the velocity addition theorem**

   a. Let us consider a first Lorentz boost in the $x > 0$ direction of normalised velocity $\beta_1 = v_1/c$, followed by a second one still in the direction $x > 0$ of normalised velocity $\beta_2 = v_2/c$. Show that the result is equivalent to a single Lorentz boost in the direction $x > 0$ of normalised velocity β_{21} given by (note this is the addition theorem for velocities)

$$\beta_{21} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}.$$

b. In order to check that the speed of light may never be reached through successive Lorentz boosts, let us consider a succession of identical Lorentz boosts all in the direction $x > 0$ and of normalised velocity $\beta = v/c$, v being a given velocity. These Lorentz boosts are applied to an inertial frame initially observed at rest, of normalised velocity $\beta_0 = 0$. After n such Lorentz boosts the frame has a normalised velocity β_n. Establish the recursion relation for successive values of β_n.

c. In order to solve these recursion relations, it proves useful to introduce a hyperbolic function parametrisation of the normalised velocities (namely exploit the hyperbolic character of the Minkowski geometry of spacetime), in terms of the rapidity ω,

$$\cosh\omega = \gamma, \quad \sinh\omega = \beta\gamma, \quad \tanh\omega = \beta, \quad \beta = \frac{v}{c}, \quad \gamma = (1-\beta^2)^{-1/2}.$$

Show then that the solution to the recursion relation is

$$\beta_n = \tanh\left(n \,\mathrm{arctanh}\, \frac{v}{c}\right).$$

d. Based on this result, study the behaviour of the asymptotic velocity, $\lim_{n\to\infty}\beta_n$.

---

a. Given the well known relations between the spacetime coordinates of a same event observed from two different inertial frames which are in relative motion at constant velocity, namely the expressions defining a Lorentz boost, one has for the first transformation of velocity β_1,

$$ct' = \gamma_1\left[ct - \beta_1 x\right], \qquad x' = \gamma_1\left[-\beta_1 ct + x\right], \tag{1}$$

and as well for the second transformation of velocity β_2,

$$ct'' = \gamma_2\left[ct' - \beta_2 x'\right], \qquad x'' = \gamma_2\left[-\beta_2 ct' + x'\right], \tag{2}$$

while the coordinates transverse to the motion remain unaffected, namely $y'' = y' = y$, $z'' = z' = z$. In these relations the following definitions apply of course,

$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}, \qquad \gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}}. \tag{3}$$

By simple substitution of the first transformation into the second, expressing thereby the composition of two colinear Lorentz boosts, one finds,

$$ct'' = \gamma_1\gamma_2(1+\beta_1\beta_2)\left[ct - \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}x\right], \qquad x'' = \gamma_1\gamma_2(1+\beta_1\beta_2)\left[-\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}ct + x\right], \tag{4}$$

a result which obviously suggests that these expressions ought to be those associated to a Lorentz boost in the x direction and of velocity given by,

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}. \tag{5}$$

However in order to establish this result, one needs to also check explicitly that the associated Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$ indeed coincides with the product $\gamma_1\gamma_2(1+\beta_1\beta_2)$. An explicit evaluation finds,

$$\gamma = (1-\beta^2)^{-1/2} = (1+\beta_1\beta_2)\left[(1+\beta_1\beta_2)^2 - (\beta_1+\beta_2)^2\right]^{-1/2} = \gamma_1\gamma_2(1+\beta_1\beta_2). \tag{6}$$

Consequently, the composition of two colinear Lorentz boosts, in the present case along the x axis, with velocities β_1 and β_2, is equivalent to a single Lorentz boost in that same direction of course, and with a velocity given by (5). A number of comments are relevant at this point.

First, one obtains a commutative or abelian composition rule, since this relation for β is symmetrical in the two initial Lorentz boosts. The reason for this fact is that both Lorentz boosts are along the same spatial direction. The situation may be compared to that of two spatial rotations: if these possess the same rotation axis, their composition is again a rotation around the same axis but with an angle simply given by the sum of the two angles of the two initial rotations. However, if the directions of the two Lorentz boosts are different, their composition defines a third Lorentz boost, but of which the direction and the velocity depend on the order in which the two Lorentz boosts are composed. Likewise for two spatial rotations around two different axes, their composition determines a third rotation which is function of the order in which the two rotations are being composed. In other words, in general, the composition of Lorentz boosts, and more generally of any Lorentz transformations including thus spatial rotations, is not commutative. It is only for common directions – whether for Lorentz boosts or spatial rotations – that the composition of Lorentz transformations is commutative, a particular case in the present Problem.

Second, one notices that the velocity composition law (5) also determines the relativistic velocity addition theorem. Indeed, referring to the expressions for that latter theorem, the same result for the velocity along the x axis is obtained. In order to obtain the addition theorem for the other two transverse components relative to the y and z axes, one would have to consider the general composition of Lorentz boosts along different directions, an exercise of interest for its own sake left to the readers' curiosity.

b. Using the result (5), clearly the sought for recursion relation is simply,

$$\beta_n = \frac{\beta_{n-1} + \beta}{1 + \beta\beta_{n-1}}, \qquad \beta_0 = 0, \qquad n = 1, 2, \ldots \tag{7}$$

In this form finding a solution to these conditions is not straightforward, unless one notices the addition theorem for the hyperbolic function $\tanh\alpha$. However if one recalls the situation for ordinary spatial rotations, one may suspect that the above relation may become linear in terms of pseudo-rotation angles related to the present Lorentz boost transformations which are characteristic of a hyperbolic geometry.

c. Considering first the situation in 1.a., let us introduce the parametrisation,

$$\cosh\alpha_1 = \gamma_1, \quad \sinh\alpha_1 = \beta_1\gamma_1, \quad \tanh\alpha_1 = \beta_1; \quad \cosh\alpha_2 = \gamma_2, \quad \sinh\alpha_2 = \beta_2\gamma_2, \quad \tanh\alpha_2 = \beta_2. \tag{8}$$

Using then the addition theorem for hyperbolic functions, the composition of two Lorentz boosts reads, in matrix form,

$$\begin{pmatrix} \cosh\alpha_2 & -\sinh\alpha_2 \\ -\sinh\alpha_2 & \cosh\alpha_2 \end{pmatrix} \begin{pmatrix} \cosh\alpha_1 & -\sinh\alpha_1 \\ -\sinh\alpha_1 & \cosh\alpha_1 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha_1+\alpha_2) & -\sinh(\alpha_1+\alpha_2) \\ -\sinh(\alpha_1+\alpha_2) & \cosh(\alpha_1+\alpha_2) \end{pmatrix}. \tag{9}$$

Consequently in the present form, it is clear that the result is the Lorentz boost in the x direction defined par the quantities,

$$\alpha = \alpha_1 + \alpha_2, \qquad \cosh\alpha = \gamma, \qquad \sinh\alpha = \beta\gamma, \qquad \tanh\alpha = \beta, \tag{10}$$

hence indeed establishing the additive (thus commutative or abelian) property of the parameter α associated to these Lorentz boosts. In particular one now recognizes in (5) the addition theorem for the function $\tanh\alpha$,

$$\tanh(\alpha_1+\alpha_2)=\frac{\tanh\alpha_1+\tanh\alpha_2}{1+\tanh\alpha_1\tanh\alpha_2}. \qquad (11)$$

Considering now the infinite succession of identical Lorentz boosts of velocity β of 1.b., it follows that one has the additive recursion relation,

$$\alpha_n=\alpha_{n-1}+\alpha,\qquad \alpha_0=0,\qquad n=1,2,\cdots, \qquad (12)$$

with of course $\beta_n=\tanh\alpha_n$ and $\beta=\tanh\alpha=v/c$. Since the solution to this relation is

$$\alpha_n=n\alpha,\qquad n=0,1,2,\ldots, \qquad (13)$$

one has for the associated velocities,

$$\beta_n=\tanh\left(n\ \mathrm{arctanh}\,\frac{v}{c}\right),\qquad n=0,1,2\ldots \qquad (14)$$

d. Using the result (14), it follows

$$\lim_{n\to\infty}\beta_n=1, \qquad (15)$$

showing how the speed of light is reached only in the limit of an infinite number of successive Lorentz boosts of identical velocity. More precisely, the speed of light is not exceeded either, since one may also write,

$$\beta_n=\frac{e^{n\alpha}-e^{-n\alpha}}{e^{n\alpha}+e^{-n\alpha}}=1-2\frac{e^{-2n\alpha}}{1+e^{-2n\alpha}}<\ 1, \qquad (16)$$

with obviously $\alpha=\mathrm{arctanh}\,v/c$. Hence for any finite value for n, the value for the velocity β_n remains strictly less that the speed of light (in vacuum), with a remainder given by the above expression.

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3. **Lorentz transformations and velocity compositions**

Consider two inertial frames R and R' sharing the same right-handed orthonormalised basis vectors $\{\hat{I}_1,\hat{I}_2,\hat{I}_3\}$, and of which the origins coincide at time $t=0=t'$. The spacetime coordinates of a same event are denoted (ct,x,y,z) with respect to the frame R and (ct',x',y',z') with respect to R'. The frame R' moves with respect to R with the constant velocity $\vec{v}_0=v_0\hat{I}_1$, with $v_0>0$. We set $\beta_0=v_0/c$ and $\gamma_0=(1-\beta_0^2)^{-1/2}$.

With respect to R, a photon of frequency ν propagates in the direction $\hat{n}$ characterised by angles (θ,φ) such that $\hat{n}=\cos\theta\,\hat{I}_1+\cos\varphi\sin\theta\,\hat{I}_2+\sin\varphi\sin\theta\,\hat{I}_3$. One wishes to determine the characteristics of the trajectory of this photon as observed from R'.

a. Given the expressions for the relevant Lorentz boost, establish the addition theorem for velocities as measured with respect to the two frames R and R'.

b. Applying this result to the photon, show that its velocity in R' is of norm c, as it should.

c. Show that with respect to the given basis vectors and the frame R' the angles (θ',φ') defined in a likewise way to the one above for the trajectory of the photon are

$$\cos\theta'=\frac{\cos\theta-\beta_0}{1-\beta_0\cos\theta},\quad \sin\theta'=\frac{1}{\gamma_0}\frac{\sin\theta}{1-\beta_0\cos\theta},\quad \tan\theta'=\frac{1}{\gamma_0}\frac{\sin\theta}{\cos\theta-\beta_0},\quad \varphi'=\varphi.$$

d. Given that for the considered photon $E = h\nu$ and $\vec{\beta} = \vec{p}c/E$, with $(E, \vec{p}c)$ defining the components of a four-vector, show that the frequency ν' of the photon in the frame R' is (**the electromagnetic Doppler effect**)

$$\nu' = \nu \, \frac{1 - \beta_0 \cos\theta}{\sqrt{1 - \beta_0^2}} = \nu \, \sqrt{\frac{(1 - \beta_0 \cos\theta)^2}{(1 - \beta_0)(1 + \beta_0)}}.$$

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a. Given the expressions for Lorentz boosts provided above, one direct method for establishing the relativistic velocity addition theorem is through composed partial differentiation with respect to the time variables in the two inertial frames involved. Since the components of the velocity vectors in each inertial frame are defined in terms of derivatives of the associated cartesian coordinates with respect to the relevant time coordinate, one first needs to identify the Jacobian factor related to the change of variable between the two time coordinates t and t'. Hence by normalising the physical dimension of the latter to units of length using the constant c of the speed of light in vacuum, which is also the physical dimension of the spatial coordinates, one directly finds,

$$\frac{d(ct')}{d(ct)} = \gamma_0 \, (1 - \beta_0 \beta_x) \, , \qquad \frac{d(ct)}{d(ct')} = \frac{1}{\gamma_0 \, (1 - \beta_0 \beta_x)}, \tag{1}$$

where $\beta_x = v_x/c$ denotes the component relative to the basis vector $\hat{I}_1$ of the velocity normalised to c of the massive point particle in the inertial frame R.

These expressions having been identified, each of the cartesian components of the velocity vector normalised to c, $\vec{\beta}' = \vec{v}\,'/c$, of the massive point particle in the inertial frame R' are given as,

$$\beta'_x = \frac{v'_x}{c} = \frac{dx'}{cdt'} = \frac{d(ct)}{d(ct')} \frac{dx'}{d(ct)} = \frac{\beta_x - \beta_0}{1 - \beta_0 \beta_x}, \tag{2}$$

$$\beta'_y = \frac{v'_y}{c} = \frac{dy'}{cdt'} = \frac{d(ct)}{d(ct')} \frac{dy'}{d(ct)} = \frac{1}{\gamma_0} \frac{\beta_y}{1 - \beta_0 \beta_x}, \tag{3}$$

$$\beta'_z = \frac{v'_z}{c} = \frac{dz'}{cdt'} = \frac{d(ct)}{d(ct')} \frac{dz'}{d(ct)} = \frac{1}{\gamma_0} \frac{\beta_z}{1 - \beta_0 \beta_x}, \tag{4}$$

thereby establishing the relativistic velocity addition theorem.

b. Using these relations the evaluation of $1 - \vec{\beta'}^2$ in terms of the components of the velocity vector normalised to c in the inertial frame R is readily completed, leading to

$$1 - \vec{\beta'}^2 = 1 - \frac{1}{(1 - \beta_0 \beta_x)^2} \left[(\beta_x - \beta_0)^2 + \frac{1}{\gamma_0^2} \left(\beta_y^2 + \beta_z^2 \right) \right], \tag{5}$$

or in terms of $1/\gamma_0^2 = 1 - \beta_0^2$,

$$1 - \vec{\beta'}^2 = \frac{(1 - \beta_0 \beta_x)^2 - (\beta_x - \beta_0)^2 - (1 - \beta_0^2)(\beta_y^2 + \beta_z^2)}{(1 - \beta_0 \beta_x)^2}. \tag{6}$$

By expanding the numerator of this expression, one finds,

$$1 - \vec{\beta'}^2 = \frac{(1 - \beta_0^2)(1 - \vec{\beta}^2)}{(1 - \beta_0 \beta_x)^2}, \tag{7}$$

as it should.

The particular case of a massless particle is distinguished by the fact that its normalised velocity vector, $\vec{\beta}$, is then always of unit norm, $\vec{\beta}^2 = 1$, and this in whatever inertial frame since such a particle always propagates at the speed of light in vacuum, c. Indeed, if $\vec{\beta}^2 = 1$ in the inertial frame R, likewise in the inertial frame R' we have the value $\vec{\beta}'^2 = 1$, given the above result. The speed of light in vacuum c is thus a relativistic invariant, and a fundamental constant of physics, related to the geometrical structure of spacetime.

c. If the normalised velocity vector $\vec{\beta}$ in the inertial frame R is represented as

$$\vec{\beta} = \beta\left[\cos\theta\hat{I}_1 + \sin\theta\cos\varphi\hat{I}_2 + \sin\theta\sin\varphi\hat{I}_3\right], \tag{8}$$

and likewise for the normalised velocity vector $\vec{\beta}'$ in the inertial frame R' by,

$$\vec{\beta}' = \beta'\left[\cos\theta'\hat{I}_1 + \sin\theta'\cos\varphi'\hat{I}_2 + \sin\theta'\sin\varphi'\hat{I}_3\right], \tag{9}$$

where in each case β et β' stand for the norms of these velocity vectors, the relations between the angles (θ', φ') and (θ, φ) in the two inertial frames R' et R are directly determined using the relativistic velocity addition theorem established above.

One readily finds for the angle φ',

$$\tan\varphi' = \frac{\beta'_z}{\beta'_y} = \frac{\beta_z}{\beta_y} = \tan\varphi, \tag{10}$$

hence,

$$\varphi' = \varphi. \tag{11}$$

The geometric interpretation of this result should be clear enough. Under a Lorentz boost only the components parallel or longitudinal with the relative motion are contracted while the transverse or perpendicular components remain unaffected. In the case of the Lorentz boost considered here along the x direction, the angles φ and φ' correspond to the angular direction relative to the basis vector $\hat{I}_2$ of the projection of the massive point particle position vector onto the plane spanned by the basis vectors $\{\hat{I}_2, \hat{I}_3\}$ which are perpendicular to the direction of motion. In other words, no relativistic contraction affects this projection, hence neither these angles, explaining why they must remain identical to one another as a purely algebraic analysis has indeed established. The geometric picture is obviously far simpler and direct.

Once the identity of these angles established, $\varphi' = \varphi$, the relation between the angles θ' and θ defined by the vectors $\vec{\beta}'$ and $\vec{\beta}$ relative to the direction $\hat{I}_1$ of the relative motion is readily obtained. For instance one has,

$$\tan\theta' = \frac{1}{\cos\varphi'}\frac{\beta'_y}{\beta'_x} = \frac{1}{\gamma_0}\frac{\beta_y}{\beta_x - \beta_0} = \frac{1}{\gamma_0}\frac{\beta\sin\theta}{\beta\cos\theta - \beta_0}. \tag{12}$$

Hence the relativistic velocity addition theorem implies such a variation, as a function of the relative velocity β_0, in the direction of the velocity vector of point particles, including massless ones for which $\beta = 1$. Incidentally note that the nonrelativistic limit $\gamma_0 \to 1$ of this expression correctly reproduces the analogous result for Galilei boost transformations between inertial frames, namely,

$$\frac{v_0}{c} \to 0\ ,\ \gamma_0 \to 1\ :\qquad \tan\theta' = \frac{v\sin\theta}{v\cos\theta - v_0}. \tag{13}$$

d. As a general fact, when expressed in the same physical units, namely those of energy for instance, the energy E and the momentum $\vec{p}c$ of a particle define a four-vector. By four-vector, one understands that these quantities, under any Lorentz transformation and in particular any Lorentz boost, transform precisely through the same relations as do the components $(x^0 = ct, \vec{x})$ of the four-vector specifying the spacetime position of a physical event. In other words, once the values for E et $\vec{p}c$ are known in some inertial frame R, those for the quantities E' et $\vec{p}\,'c$ in some other inertial frame R' are directly identified using the relations defining Lorentz transformations.

In the particular case of the photon which is massless, we have furthermore that $E = |\vec{p}|c$, while the physical quantum character of that particle implies that its energy is given by $E = h\nu = |\vec{p}|c$, h being Planck's constant of quantum physics. Considering then the component $p_x c = |\vec{p}|c\cos\theta = h\nu\cos\theta$ of the momentum of a photon propagating in the direction of angle θ relative to the basis vector $\hat{I}_1$, its energy $E' = h\nu'$ in the inertial frame R' is thus given by the time component of the corresponding Lorentz boost transformation,

$$h\nu' = E' = \gamma_0\left[E - \beta_0 p_x c\right] = \gamma_0\left[h\nu - \beta_0 h\nu\cos\theta\right], \tag{14}$$

leading to the relation,

$$\nu' = \nu\gamma_0\left[1 - \beta_0\cos\theta\right] = \nu\frac{1 - \beta_0\cos\theta}{\sqrt{1-\beta_0^2}} = \nu\sqrt{\frac{(1-\beta_0\cos\theta)^2}{(1-\beta_0)(1+\beta_0)}}. \tag{15}$$

This result expresses in general terms the Doppler effect for electromagnetic radiation, having used the fact that such radiation is in fact composed of relativistic and quantum massless particles, namely photons.

Finally, if the photon does not propagate in the direction of the basis vector $\hat{I}_1$ in the inertial frame R', this translates in the value $\cos\theta' = 0$, or equivalently $\beta'_x = 0$. Given the relativistic velocity addition theorem, such a configuration implies that the relative velocity of the two inertial frames is precisely such that $\beta_0 = \cos\theta$, hence providing the direction of propagation of the photon in the inertial frame R. Using this information in the formula for the Doppler effect, it follows that

$$\nu' = \nu\sqrt{1-\beta_0^2}. \tag{16}$$

It is intriguing to try imagine what happens to such a photon in the nonrelativistic limit $\beta_0 \to 1$.

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4. Given a unit of mass m_0, the relativistic energy and momentum of a free particle are measured to be

$$E = 5\,m_0c^2, \qquad \vec{p} = 4\,m_0 c\,\hat{x},$$

$\hat{x}$ being a normalised direction in space.

a. What is the mass m_1 of the particle?

b. What is the norm v of the velocity of the particle?

c. If τ_{proper} is the proper lifetime of the particle, what is its lifetime τ in the inertial frame in which it is observed with the above values of E and $\vec{p}$?

d. The particle of mass m_1 suddenly collides in an elastic scattering with another particle of unknown mass m_2 initially at rest. After the collision the latter particle is observed in the $\hat{x}$ direction with a velocity of norm $v_2 = \frac{3}{5}\,c$. What is the value of its mass m_2?

e. After the collision, what is the velocity $\vec{v}_1$ of the incoming particle?

The relations central to solving this type of questions are the expressions for the relativistic energy and momentum of a point particle of masse m,

$$E = mc^2\gamma, \qquad \vec{p}c = mc^2\gamma\vec{\beta}, \qquad \vec{\beta} = \frac{\vec{p}c}{E}, \qquad E^2 - (\vec{p}c)^2 = (mc^2)^2,$$

where the last of these relations provides the relativistic invariant associated to the four-vector $(E, \vec{p}c)$, while one has of course,

$$\vec{\beta} = \frac{\vec{v}}{c}, \qquad \gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}}.$$

a. Determining the relativistic invariant provides directly the mass m_1 in units of the mass m_0,

$$E^2 - (\vec{p}c)^2 = \left[5^2 - 4^2\right](m_0c^2)^2 = \left(3m_0c^2\right)^2.$$

Hence,

$$m_1 = 3m_0 \ .$$

b. The velocity of the particle is simply given by the ratio $\vec{p}c/E$, namely $\vec{\beta} = \frac{4}{5}\hat{x}$. Hence the norm of that velocity is,

$$v = \frac{4}{5}c.$$

c. Therefore the Lorentz factor γ of the particle is,

$$\gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}} = \frac{5}{3}.$$

Note that using the value $E = 5m_0c^2$, knowledge of the factor γ also allows to conclude that $m_1 = 3m_0$. The facteur γ also determines the time dilation factor. Hence the mean lifetime of the particle in motion is,

$$\tau = \gamma\tau_{\text{proper}} = \frac{5}{3}\tau_{\text{proper}}.$$

d. Knowing the velocity of the particle of mass m_2 after the collision, the corresponding relativistic factors are thus,

$$\beta_2 = \frac{3}{5}, \qquad \gamma_2 = \frac{1}{\sqrt{1 - \beta_2^2}} = \frac{5}{4}, \qquad \beta_2\gamma_2 = \frac{3}{4}.$$

It now suffices to consider the conservation of total relativistic energy and momentum for the elastic collision, namely

$$5m_0c^2 + m_2c^2 = \sqrt{(p_1c)^2 + (3m_0c^2)^2} + \frac{5}{4}m_2c^2, \qquad 4m_0c^2 = p_1c + \frac{3}{4}m_2c^2,$$

where in the second condition (that of total relativistic momentum conservation), p_1c is the **component** along the $\hat{x}$ direction of the relativistic momentum of the particle of masse m_1 after the collision, while this same quantity is used already in the first energy conservation condition to express the relativistic energy of that same particle. It now suffices to substitute the second relation for p_1c into the first equation, and to solve for the second unknown, m_2, to find the value,

$$m_2 = 7m_0.$$

Indeed, the first relation also writes as,

$$\sqrt{\left(4m_0c^2 - \frac{3}{4}m_2c^2\right)^2 + 9(m_0c^2)^2} = 5m_0c^2 - \frac{1}{4}m_2c^2.$$

Taking the square of this last identity, one has,

$$16(m_0c^2)^2 - 6(m_0c^2)(m_2c^2) + \frac{9}{16}(m_2c^2)^2 + 9(m_0c^2)^2 = 25(m_0c^2)^2 - \frac{5}{2}(m_0c^2)(m_2c^2) + \frac{1}{16}(m_2c^2)^2,$$

namely,

$$\frac{1}{2}(m_2c^2)^2 = \frac{7}{2}(m_0c^2)(m_2c^2).$$

e. Using the last result, it follows that

$$p_1c = \left[4 - \frac{3}{4}7\right] m_0c^2 = -\frac{5}{4}m_0c^2,$$

hence

$$E_1 = m_0c^2\sqrt{\frac{25}{16} + 9} = \frac{13}{4}m_0c^2,$$

and finally,

$$\vec{v}_1 = \frac{\vec{p}_1c}{E_1} = -\frac{5}{13}\,c\,\hat{x}.$$

Note that after the collision, the particle of mass $m_1 = 3m_0$ recoils in a direction opposite to the initial direction, as it should since $m_2 = 7m_0$ is larger than m_1.

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5. Let us consider the disintegration

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu$$

with the mass values $m_{\pi^+}c^2 = 139.57$ MeV, $m_{\mu^+}c^2 = 105.66$ MeV and $m_{\nu_\mu}c^2 = 0$ MeV, the initial particle being at rest.

a. What is the neutrino's energy E_{ν_μ} in the final state?

b. What is the μ^+'s energy E_{μ^+} in the final state?

c. In units of c, what is the velocity of the μ^+?

d. In units of c, what is the velocity of the neutrino ν_μ?

Obviously the solution to the Problem relies on the conservation laws for the total relativistic energy and momentum of the system. Let us recall that for a particle of arbitrary mass m, one always has

$$E^2 - (\vec{p}c)^2 = \left(mc^2\right)^2, \tag{1}$$

where E and $\vec{p}_c$ are, respectively, its relativistic energy and momentum, with in particular E including its rest-mass energy mc^2 in the massive case. Furthermore for a massive particle one has,

$$E = mc^2\,\gamma, \qquad \vec{p}c = mc^2\,\gamma\,\vec{\beta}, \qquad \vec{\beta} = \frac{\vec{p}c}{E}, \tag{2}$$

where of course,

$$\vec{\beta} = \frac{\vec{v}}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\vec{\beta}^{\,2}}}, \qquad \beta\,\gamma = \sqrt{\gamma^2-1}, \tag{3}$$

$\vec{v}$ being the velocity of the particle, while for a massless particle,

$$m = 0\ :\qquad E = |\vec{p}c|, \qquad \beta = \frac{|\vec{p}c|}{E} = 1. \tag{4}$$

a. In the case of the considered disintegration, and within the proper inertial frame of the decaying particle, the kinematic relation of the conservation of the total relativistic energy is thus,

$$m_\mu c^2\gamma\,+\,E_{\nu_\mu} = m_\pi c^2, \tag{5}$$

γ being the Lorentz factor determined by the velocity of the μ^+, and E_{ν_μ} being the energy of the produced neutrino ν_μ.

Furthermore since the total initial momentum is identically vanishing for that choice of inertial frame, the momenta of the two particles in the final state are opposite to one another and of equal norms. The kinematic relation expressing the conservation of the total relativistic momentum thus translates in terms of the following second equation, it being understood that the mass of the neutrino is taken here to be vanishing for all practical purposes,

$$m_\mu c^2\,\beta\gamma = E_{\nu_\mu}, \tag{6}$$

β being the μ^+ velocity normalised to the speed of light in vacuum, c. Direct substitution of this second equation into the first, and using the identity $(1+\beta)\gamma = \sqrt{\frac{1+\beta}{1-\beta}}$, one finds,

$$m_\mu c^2(1+\beta)\gamma = m_\mu c^2\sqrt{\frac{1+\beta}{1-\beta}} = m_\pi c^2, \qquad \sqrt{\frac{1+\beta}{1-\beta}} = \frac{m_\pi c^2}{m_\mu c^2} \equiv r, \tag{7}$$

hence finally,

$$\beta = \frac{r^2-1}{r^2+1} = \frac{(m_\pi c^2)^2-(m_\mu c^2)^2}{(m_\pi c^2)^2+(m_\mu c^2)^2}. \tag{8}$$

Consequently one also has,

$$1-\beta^2 = \frac{4r^2}{(r^2+1)^2}, \qquad \gamma = \frac{r^2+1}{2r}, \qquad \beta\gamma = \frac{r^2-1}{2r}. \tag{9}$$

Therefore, the energy of the final neutrino is given as,

$$E_{\nu_\mu} = m_\mu c^2\beta\gamma = \frac{(m_\pi c^2)^2-(m_\mu c^2)^2}{2(m_\pi c^2)} = 29.791\ \text{MeV}. \tag{10}$$

b. Given the above kinematic solution, the relativistic energy of the final muon is obtained as $E_{\mu^+} = m_\pi c^2 - E_{\nu_\mu}$, namely,

$$E_{\mu^+} = m_\mu c^2\,\gamma = \frac{(m_\pi c^2)^2+(m_\mu c^2)^2}{2(m_\pi c^2)} = 109.78\ \text{MeV}. \tag{11}$$

c. Likewise, the velocity β of the final muon takes the value,

$$\beta = \frac{v_{\mu^+}}{c} = \frac{(m_\pi c^2)^2 - (m_\mu c^2)^2}{(m_\pi c^2)^2 + (m_\mu c^2)^2} = 0.2714. \qquad (12)$$

d. And finally since the neutrino mass is taken to be vanishing in the above kinematic solution, the neutrino is produced with exactly the speed of light in vacuum,

$$\frac{v_{\nu_\mu}}{c} = 1.00. \qquad (13)$$

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6. **Compton scattering**
Consider the Compton scattering of a photon on an electron initially at rest in the inertial frame of the laboratory,

$$\gamma + e^- \longrightarrow \gamma + e^-.$$

Let E_γ and $\vec{p}_\gamma$ be the energy and momentum of the incoming photon, and E'_γ and $\vec{p}_\gamma{}'$ be those of the photon in the final state scattered in the angular direction θ with respect to the incoming photon. The electron mass is denoted m_e.

a. Show that as a function of the scattering angle the final photon energy is

$$E'_\gamma = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e c^2}(1 - \cos\theta)}.$$

b. Using the quantum relation between the energy and the wavelength of a photon in vacuum, establish the Compton effect

$$\lambda'_\gamma - \lambda_\gamma = \lambda_e \left(1 - \cos\theta\right),$$

λ'_γ and λ_γ being the initial and final photon wavelengths, respectively, and λ_e the Compton wavelength of the electron

$$\lambda_e = \frac{2\pi\hbar c}{m_e c^2} = \frac{hc}{m_e c^2}.$$

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a. In terms of the notations specified above, the conservation conditions for the total relativistic energy and momentum of the scattering write as, respectively,

$$E_\gamma + m_e c^2 = E_\gamma{}' + E_e{}', \qquad \vec{p}_\gamma c = \vec{p}_\gamma{}'c + \vec{p}_e{}'c, \qquad (1)$$

$E_e{}'$ et $\vec{p}_e{}'c$ being, respectively, the (relativistic) energy and momentum of the electron in the final state (these quantities for the electron in the initial state take of course the values $m_e c^2$ and a vanishing momentum, respectively, in the laboratory inertial frame). One also has,

$$E_\gamma = |\vec{p}_\gamma c|, \qquad E_\gamma{}' = |\vec{p}_\gamma{}'c|, \qquad E_e{}' = \sqrt{(\vec{p}_e{}'c)^2 + (m_e c^2)^2}, \qquad (2)$$

for these particles of zero mass – the photon – and m_e – the electron.

From the above vector relation it follows that,

$$\vec{p}_e{}'c = \vec{p}_\gamma c - \vec{p}_\gamma{}'c. \qquad (3)$$

Taking the scalar product of this identity with itself in which the momenta of the initial and final photons have been brought together on one side of the relation, one derives an expression in terms of the photon scattering angle θ,

$$E_e'^2 - \left(m_e c^2\right)^2 = E_\gamma{}^2 + E_\gamma'^2 - 2E_\gamma\, E_\gamma{}'\cos\theta. \tag{4}$$

Using then the (scalar) relation for the conservation of energy in the process, written as,

$$E_e{}' = E_\gamma - E_\gamma{}' + m_e c^2, \tag{5}$$

and then the square of this latter identity, one also finds,

$$E_e'^2 - \left(m_e c^2\right)^2 = E_\gamma{}^2 + E_\gamma'^2 - 2E_\gamma E_\gamma{}' + 2m_e c^2\left(E_\gamma - E_\gamma{}'\right). \tag{6}$$

The difference of (4) and (6) then produces,

$$E_\gamma{}'\left[E_\gamma(1-\cos\theta) + m_e c^2\right] = m_e c^2\, E_\gamma, \tag{7}$$

and finally,

$$E_\gamma{}' = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e c^2}(1-\cos\theta)}. \tag{8}$$

b. The frequency ν and the energy E_γ of a photon obey the quantum relation,

$$E_\gamma = h\nu, \tag{9}$$

while for a photon propagating in vacuum, its frequency ν and wavelength λ are such that

$$\lambda = \frac{c}{\nu}. \tag{10}$$

Therefore one simply has,

$$\lambda = \frac{hc}{E_\gamma} = \frac{2\pi\hbar c}{E_\gamma}. \tag{11}$$

Given the result (8) for Compton scattering, and with λ_e being the electron Compton wavelength, one thus obtains

$$\lambda'_\gamma = \frac{2\pi\hbar c}{E_\gamma{}'} = \frac{2\pi\hbar c}{E_\gamma}\left[1 + \frac{E_\gamma}{m_e c^2}(1-\cos\theta)\right] = \lambda_\gamma + \lambda_e\,(1-\cos\theta)\,, \tag{12}$$

hence finally the celebrated Compton relation,

$$\lambda'_\gamma - \lambda_\gamma = \lambda_e\,(1-\cos\theta)\,. \tag{13}$$

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7. Let us consider the relativistic equation of motion of a particle of mass m subjected to a constant force $\vec{F} = F\,\hat{F}$,

$$\frac{d\vec{p}(t)}{dt} = \vec{F}, \qquad \vec{p}(t) = \frac{m\vec{v}(t)}{\sqrt{1 - \frac{\vec{v}^{\,2}(t)}{c^2}}},$$

given the initial conditions $\vec{x}(t=0) = \vec{x}_0$ and $\vec{p}(t=0) = \vec{p}_0$. In the nonrelativistic limit of Newton's mechanics the trajectory is of constant acceleration and generally parabolic (unless the initial data $\vec{x}_0$ and $\vec{p}_0$ are colinear, or one at least is vanishing, in which case the trajectory is linear and of constant acceleration of course). What is the situation in the relativistic context?

a. Show that the solution for the velocity is

$$\vec{v}(t) = \frac{1}{m} \frac{\vec{F}t + \vec{p}_0}{\sqrt{1 + \frac{1}{m^2c^2}\left(\vec{F}t + \vec{p}_0\right)^2}}.$$

How does the norm of this velocity evolve in time?

b. Given the particular initial conditions $\vec{p}_0 = \vec{0}$ and $\vec{x}_0 = \vec{0}$, show that the trajectory is given as

$$\vec{v}(t) = \frac{1}{m} \frac{\vec{F}t}{\sqrt{1 + \frac{F^2t^2}{m^2c^2}}}, \qquad \vec{x}(t) = \frac{mc^2}{F}\,\hat{F}\left[\sqrt{1 + \frac{F^2t^2}{m^2c^2}} - 1\right].$$

Consider the graphs of $|\vec{v}(t)|$ and $|\vec{x}(t)|$ in order to observe the transition from the nonrelativistic regime to the relativistic one.

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a) The applied force being constant, the solution to the equation of motion

$$\frac{d\vec{p}(t)}{dt} = \vec{F}, \tag{1}$$

is obviously

$$\vec{p}(t) = \vec{F}t + \vec{p}_0, \tag{2}$$

$\vec{p}_0$ being the initial value for the relativistic momentum. Given the definition of the latter quantity recalled above, it follows that,

$$\vec{v}(t) = \sqrt{1 - \frac{\vec{v}^{\,2}(t)}{c^2}}\,\frac{1}{m}\left[\vec{F}t + \vec{p}_0\right]. \tag{3}$$

Taking the scalar product of this expression with itself, there follows a simple linear relation for $\vec{v}^{\,2}(t)$, of which the solution is

$$\vec{v}^{\,2}(t) = \frac{A}{1 + A/c^2}, \qquad A = \frac{1}{m^2}\left[\vec{F}t + \vec{p}_0\right]^2. \tag{4}$$

Once substitute in the above expression for $\vec{v}(t)$, one finally establishes

$$\vec{v}(t) = \frac{1}{m} \frac{\vec{F}t + \vec{p}_0}{\sqrt{1 + \frac{1}{m^2c^2}(\vec{F}t + \vec{p}_0)^2}}. \tag{5}$$

In the limit of an infinite time, one thus observes generally that,

$$\lim_{t\to\infty} \vec{v}(t) = c\,\hat{F}, \tag{6}$$

showing that the point particle never exceeds the speed of light in vacuum, and does not reach that value in any finite time interval. In the limit of an infinite time interval its motion is in the direction $\hat{F}$ of the applied force, the contribution of the initial momentum $\vec{p}_0$ having then become totally negligible in comparison to that of the force, $\vec{F}t$.

Furthermore in the nonrelativistic regime where $(\vec{F}t + \vec{p}_0)^2 \ll m^2c^2$, the velocity is given by the nonrelativistic solution for a motion of constant acceleration $\vec{F}/m$ associated to the constant force $\vec{F}$, as it should,

$$\vec{v}(t) \simeq \frac{1}{m}\left[\vec{F}t + \vec{p}_0\right]. \tag{7}$$

b) In the specific case – of course chosen to simplify the analysis – that the initial conditions are $\vec{p}_0 = \vec{0}$ and $\vec{x}_0 = \vec{0}$, one has,

$$\frac{d\vec{x}(t)}{dt} = \vec{v}(t) = \frac{F}{m} \frac{t}{\sqrt{1 + \frac{F^2 t^2}{m^2 c^2}}} \hat{F}, \tag{8}$$

of which the solution obviously is,

$$\vec{x}(t) = \frac{mc^2}{F} \left[\sqrt{1 + \frac{F^2 t^2}{m^2 c^2}} - 1 \right] \hat{F}. \tag{9}$$

Consequently, since there is then as distinguished spatial direction only that of the applied constant force, $\hat{F}$, the position, $\vec{x}(t)$, and velocity, $\vec{v}(t)$, vectors are parallel to that direction. However, even though the solution is a straight trajectory, it is not one of constant acceleration, although this is the case in the nonrelativistic limit. The point particle of mass m is initially at rest, and being subjected to the constant force $\vec{F}$, is set into motion in the direction of that force. For as long as its velocity remains, in norm, much less than the speed of light in vacuum, c, the system is in a nonrelativistic regime for which Newton's laws of motion remain valid, and indeed, given the above exact solution, one then finds,

$$\vec{x}(t) \stackrel{t \simeq 0}{\simeq} \frac{1}{2} \frac{\vec{F}}{m} t^2, \qquad \vec{v}(t) \stackrel{t \simeq 0}{\simeq} \frac{\vec{F}}{m} t. \tag{10}$$

However since in the limit of an infinite time interval the point particle reaches a speed comparable to that of light in vacuum, one finds in the relativistic regime,

$$\vec{x}(t) \stackrel{t \to \infty}{\simeq} ct\,\hat{F}, \qquad \vec{v}(t) \stackrel{t \to \infty}{\simeq} c\,\hat{F}. \tag{11}$$

In other words, if one considers the graph of the function $|\vec{x}(t)|$, this graph includes two components. First a parabolic component for small t values – the time scale being set by the value of mc/F–, namely $t \ll mc/F$, corresponding to the nonrelativistic solution. Next, a linear component of uniform motion at speed c for large values of t, $t \gg mc/F$, corresponding to the relativistic regime of the system. For intermediate values of t, $t \simeq mc/F$, the solution is given by the above exact expressions. Likewise for the time dependence of the speed of the particle, $|\vec{v}(t)|$, one has a motion of constant acceleration in the nonrelativistic regime $t \ll mc/F$, a motion of essentially null acceleration in the relativistic regime $t \gg mc/F$ and a speed essentially constant and equal to c, and otherwise an intermediate velocity regime for $t \simeq mc/F$ described by the above exact expression. In particular, note that the particle never reaches the speed of light in any finite time interval, while its energy increases without limit towards an infinite value after an infinite time interval.

This simple example thus illustrates how Einstein's mechanics of Special Relativity extends into the regime of velocities comparable to the speed of light in vacuum the laws of Newton's mechanics for point particles.

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8. Show that the covariant equation of motion of a massive particle in a background electromagnetic field, namely

$$mc^2 \frac{du_\mu}{ds} = qc\, F_{\mu\nu}\, u^\nu, \quad ds = cd\tau, \quad u^\mu = \frac{dx^\mu}{ds},$$

with τ being the proper time, is equivalent to the Lorentz force equation.

Given our choice of signature for the Minkowski metric, $\eta_{\mu\nu} = \text{diag}\,(+\,-\,-\,-)$, we have

$$u^\mu = \frac{dx^\mu}{ds} = \frac{dx^\mu}{d(c\tau)} = \gamma\left(1, \vec{\beta}\right) = \frac{1}{mc^2}\left(E, \vec{p}c\right) = \frac{1}{mc^2}\,P^\mu, \tag{1}$$

since $x^\mu = (ct, \vec{x})$ and $t = \gamma\tau$. Consequently,

$$mc^2 \frac{du_\mu}{ds} = \frac{dP_\mu}{d(c\tau)} = \frac{d}{d\tau}\left(\frac{E}{c}, -\vec{p}\right) = \gamma\frac{d}{dt}\left(\frac{E}{c}, -\vec{p}\right). \tag{2}$$

On the other hand, we have

$$F_{\mu\nu} = \partial_\mu A_\nu \,-\, \partial_\nu A_\mu. \tag{3}$$

In terms of components,

$$F_{0i} = \frac{\partial A_i}{\partial x^0} \,-\, \frac{\partial A_0}{\partial x^i} = -\frac{\partial A^i}{\partial(ct)} \,-\, \frac{\partial A^0}{\partial x^i} = \frac{E^i}{c}, \tag{4}$$

since $A^\mu = (\frac{\Phi}{c}, \vec{A})$ and $\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}$. Similarly

$$F_{ij} = \partial_i A_j \,-\, \partial_j A_i = -\frac{\partial A^j}{\partial x^i} \,+\, \frac{\partial A^i}{\partial x^j} = -\epsilon^{ijk}\,B^k, \tag{5}$$

since $\vec{B} = \vec{\nabla}\times\vec{A}$.

Consequently, for the time component of the proposed equation of motion one finds

$$qcF_{0\nu}\,u^\nu = qcF_{0i}\,u^i = qc\frac{1}{c}E^i\,\frac{p^i c}{mc^2} = \frac{q}{mc}E^i p^i = \gamma\frac{q}{c}\,v^i\,E^i = \gamma\frac{q}{c}\dot{\vec{r}}\cdot\vec{E}, \tag{6}$$

while for the space components,

$$qcF_{i\nu}u^\nu = qcF_{i0}u^0 + qcF_{ij}u^j = -qc\frac{E^i}{c}\frac{E}{mc^2} - qc\epsilon^{ijk}B^k\frac{p^j c}{mc^2} = -q\gamma E^i \,-\, q\gamma\epsilon^{ijk}v^j B^k. \tag{7}$$

Combining then all these expressions in the proposed equation of motion in covariant form

$$mc^2\frac{du_\mu}{ds} = qcF_{\mu\nu}u^\nu, \tag{8}$$

the time component translates into

$$\gamma\frac{d}{dt}\frac{E}{c} = \gamma\frac{q}{c}v^i E^i, \qquad \frac{dE}{dt} = q\dot{\vec{r}}\cdot\vec{E}, \tag{9}$$

while for the space components

$$-\gamma\frac{d\vec{p}}{dt} = -\gamma\left(q\vec{E} + q\dot{\vec{r}}\times\vec{B}\right), \qquad \frac{d\vec{p}}{dt} = q\vec{E} + q\dot{\vec{r}}\times\vec{B}. \tag{10}$$

This last equation is seen to be the usual Lorentz force equation of motion for the relativistic momentum $\vec{p} = m\vec{v}/\sqrt{1-\vec{v}^{\,2}/c^2}$, while the former equation in fact follows from it by projection onto the velocity vector $\dot{\vec{r}}$. Indeed we know that

$$\dot{\vec{r}}\cdot\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}}\right) = \frac{dE}{dt}, \tag{11}$$

while the Lorentz force projected onto $\dot{\vec{r}}$ reduces to the power developed by the electric field, $q\dot{\vec{r}} \cdot \vec{E}$. In conclusion, the equation

$$mc^2 \frac{du_\mu}{ds} = qc\, F_{\mu\nu}\, u^\nu \tag{12}$$

is indeed the manifestly spacetime covariant form of the Lorentz force equation of motion for the electromagnetic coupling of a charged massive point particle to the electromagnetic field components $\vec{E}$ and $\vec{B}$, namely the field strength tensor $F_{\mu\nu}$.