An Introduction to Quantum Physics
and
Relativistic Quantum Field Theory

AIMS Lectures: 24 January – 11 February 2011
Assignment 2

Example of Solutions

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1. **The Two Dimensional Spherically Symmetric Quantum Harmonic Oscillator**
Consider the Lagrange function for the two dimensional spherically symmetric harmonic oscillator of mass m and angular frequency ω in the euclidean plane of cartesian coordinates $x_i(t)$ $(i = 1, 2)$,

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}m\omega^2\left(x_1^2 + x_2^2\right).$$

1.1. Establish the Hamiltonian formulation of this system in terms of its canonical phase space coordinates $(x_1, p_1; x_2, p_2)$.

1.2. Apply the rules of canonical quantisation in order to define the quantum Hamiltonian operator $\hat{H}$ and the commutation relations of the basic operators $\hat{x}_i$ and $\hat{p}_i$ at time $t = 0$.

1.3. Introduce the operators

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right), \qquad a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right), \qquad i = 1, 2,$$

and next

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp i a_2\right), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm i a_2^\dagger\right).$$

Show that each of the sets of operators $(a_i, a_i^\dagger)$ and $(a_\pm, a_\pm^\dagger)$ define two commuting Fock algebras,

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}\mathbb{I}, \qquad \left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}.$$

1.4. Express the quantum Hamiltonian $\hat{H}$ in terms of the annihilation and creation operators $a_\pm$ and $a_\pm^\dagger$, and show how its eigenspectrum may then be constructed in terms of the Fock states associated to the Fock algebra $(a_\pm, a_\pm^\dagger)$.

1.5. Draw in a diagram of which the horizontal axis is the variable $(n_+ - n_-)$ and the vertical one the energy value, the spectrum of energy states, where n_+ (resp., n_-) is the eigenvalue of the number operator $a_+^\dagger a_+$ (resp., $a_-^\dagger a_-$). Describe the degeneracies that exist for this spectrum.

1.6. Consider the angular momentum of the system, $L = m\left(x_1\dot{x}_2 - x_2\dot{x}_1\right)$, and express it as a function of the phase space variables. Identify the corresponding quantum operator, and finds its expression in terms of the Fock operators $(a_\pm, a_\pm^\dagger)$. Determine the eigenspectrum of the quantum angular momentum $\hat{L}$, namely the spectrum of its eigenvalues and eigenstates.

1.7. In the Heisenberg picture, solve for the time dependence of the operators $a_\pm$ and $a_\pm^\dagger$ and identify the time dependence of the position observables $\hat{x}_i(t)$ $(i = 1, 1)$. Infer then what the solutions to the dynamics are for the classical coordinates $x_i(t)$ $(i = 1, 2)$.

1.1. The Lagrange function L of the two dimensional spherically symmetric harmonic oscillator being of the form $L = T - V$, with T (resp., V) the kinetic (resp., potential) energy, as we know the Hamiltonian is given as $H = T + V$, where the kinetic energy is now expressed in terms of the momenta, p_1 and p_2, conjugate to the cartesien degrees of freedom, x_1 and x_2, with

$$p_1 = m\dot{x}_1, \qquad p_2 = m\dot{x}_2, \qquad T = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2. \tag{1}$$

Consequently the Hamiltonian formulation of this system consists of the four dimensional phase space spanned by the variables $(x_1, p_1; x_2, p_2)$ which are canonical phase space coordinates with the canonical Poisson brackets,

$$\{x_1, p_1\} = 1 = \{x_2, p_2\}, \tag{2}$$

while the Hamiltonian reads

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}m\omega^2\left(x_1^2 + x_2^2\right). \tag{3}$$

Clearly a detailed analysis starting from the Lagrange function and the usual definitions,

$$p_i = \frac{\partial L}{\partial \dot{x}_i}, \quad i = 1, 2, \qquad H = \dot{x}_i p_i - L, \tag{4}$$

reproduces precisely the above Hamiltonian formulation.

1.2. According to the rules of canonical quantisation, the space of quantum states of the quantum system is the[1] representation of the Heisenberg algebra

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathbb{I}, \qquad \hat{x}_i^\dagger = \hat{x}_i, \qquad \hat{p}_i^\dagger = \hat{p}_i, \tag{5}$$

while the quantum Hamiltonian is given by the following operator, in which no ordering ambiguity arises,

$$\hat{H}_H = \frac{1}{2m}\left(\hat{p}_1^2 + \hat{p}_2^2\right) + \frac{1}{2}m\omega^2\left(\hat{x}_1^2 + \hat{x}_2^2\right). \tag{6}$$

All these conditions and operators are defined at the reference time $t = 0$, whether in the Schrödinger or the Heisenberg picture of quantum mechanics.

1.3. Clearly, since the Lagrange function of the system is simply the sum of the Lagrange functions of two decoupled one dimensional harmonic oscillators sharing the same angular frequency ω, the Hamiltonian of the system is also the sum of the Hamiltonians of two such oscillators. Consequently, in order to diagonalise the total Hamiltonian above, $\hat{H}$, it is appropriate to introduce the associated Fock algebras of creation and annihilation operators for each of the cartesian degrees of freedom, $\hat{x}_i$, and their conjugate momenta, $\hat{p}_i$ ($i = 1, 2$), including the normalisation factors as they apply for the one dimensional case. Hence

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right), \qquad a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right), \qquad i = 1, 2, \tag{7}$$

while the inverse relations read

$$\hat{x}_i = \sqrt{\frac{\hbar}{2m\omega}}\left(a_i + a_i^\dagger\right), \qquad \hat{p}_i = -im\omega\sqrt{\frac{\hbar}{2m\omega}}\left(a_i - a_i^\dagger\right), \qquad i = 1, 2. \tag{8}$$

In particular, it follows from the above tensor product of the two Heisenberg algebras in the operators $(\hat{x}_i, \hat{p}_i)$ ($i = 1, 2$), that the operators $(a_i, a_i^\dagger)$ ($i = 1, 2$) obey a tensor product of two commuting Fock algebras,

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}\mathbb{I}, \qquad i, j = 1, 2. \tag{9}$$

[1]Indeed, in the case of the Heisenberg algebra over euclidean space, up to unitary transformations its representation is unique.

A direct substitution in the Hamiltonian operator then finds

$$\begin{aligned}\hat{H} &= \frac{1}{2}\hbar\omega\left(a_1^\dagger a_1 + a_1 a_1^\dagger + a_2^\dagger a_2 + a_2 a_2^\dagger\right)\\ &= \hbar\omega\left(a_1^\dagger a_1 + \frac{1}{2} + a_2^\dagger a_2 + \frac{1}{2}\right)\\ &= \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right),\end{aligned} \tag{10}$$

where in the second line of these expressions one recognizes of course the sum of the Hamiltonians of two harmonic oscillators in the cartesian coordinates x_1 and x_2. Consequently, the Hamiltonian operator is readily diagonalised on the Fock state basis associated to the Fock algebras $(a_i, a_i^\dagger)$ $(i = 1, 2)$.

However, since the classical trajectories of the spherically symmetric oscillator in the (x_1, x_2) configuration space are generically closed ellipses, with a certain value for the conserved angular-momentum which may be either positive or negative (or vanishing), it is relevant to consider a combination of the degrees of freedom x_1 and x_2 in terms of the complex variables $z_\pm = x_1 \pm i x_2$, with $z_\pm^* = z_\mp$, of which the complex phase has the same sign as that of the angular-momentum. This is the main reason why we are invited to introduce now a new basis of Fock algebras defined by the operators

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp i a_2\right), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm i a_2^\dagger\right). \tag{11}$$

Inverting these relations, one has

$$a_1 = \frac{1}{\sqrt{2}}\left(a_+ + a_-\right), \quad a_2 = \frac{i}{\sqrt{2}}\left(a_+ - a_-\right), \quad a_1^\dagger = \frac{1}{\sqrt{2}}\left(a_+^\dagger + a_-^\dagger\right), \quad a_2^\dagger = -\frac{i}{\sqrt{2}}\left(a_+^\dagger - a_-^\dagger\right), \tag{12}$$

hence finally,

$$\begin{aligned}\hat{x}_1 &= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ + a_- + a_+^\dagger + a_-^\dagger\right) \quad , \qquad \hat{x}_2 = \frac{i}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ - a_- - a_+^\dagger + a_-^\dagger\right),\\ \hat{p}_1 &= -i\frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ + a_- - a_+^\dagger - a_-^\dagger\right) \quad , \qquad \hat{p}_2 = \frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ - a_- + a_+^\dagger - a_-^\dagger\right).\end{aligned} \tag{13}$$

Note that we have

$$\hat{z}_\pm = \hat{x}_1 \pm i\hat{x}_2 = \sqrt{\frac{\hbar}{m\omega}}\left(a_\mp + a_\pm^\dagger\right), \tag{14}$$

hence the name of "helicity" basis of the Fock algebra for the operators $(a_\pm, a_\pm^\dagger)$.

A simple calculation then finds that the sets of operators $(a_+, a_+^\dagger)$ and $(a_-, a_-^\dagger)$ commute with one another, while each pair of operators once again defines a Fock algebra,

$$\left[a_+, a_+^\dagger\right] = \mathbb{I} = \left[a_-, a_-^\dagger\right]. \tag{15}$$

1.4. Given the above relations between the different linear redefinitions of operators, it only takes a line of calculation to find out that the total quantum Hamiltonian of the system, expressed in terms of the helicity operators $(a_\pm, a_\pm^\dagger)$, simply reads as

$$\hat{H} = \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right). \tag{16}$$

Consequently, this operator is diagonalised also on the Fock basis of the eigenstates of the two number operators $N_\pm = a_\pm^\dagger a_\pm$,

$$N_+|n_+, n_-\rangle = n_+|n_+, n_-\rangle, \qquad N_-|n_+, n_-\rangle = n_-|n_+, n_-\rangle, \tag{17}$$

namely the states defined by

$$|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!\, n_-!}}\left(a_+^\dagger\right)^{n_+}\left(a_-^\dagger\right)^{n_-}|0, 0\rangle, \qquad n_+, n_- = 0, 1, 2, \ldots, \tag{18}$$

$|0,0\rangle$ being the Fock vacuum of the system, such that $a_\pm|0,0\rangle = 0$ and $\langle 0,0|0,0\rangle = 1$. These Fock states are orthonormalised,

$$\langle n_+, n_-|m_+, m_-\rangle = \delta_{n_+,m_+}\delta_{n_-,m_-}. \tag{19}$$

Consequently, the energy eigenspectrum of the two dimensional spherically symmetric harmonic oscillator is simply obtained as

$$\hat{H}|n_+, n_-\rangle = E(n_+, n_-)|n_+, n_-\rangle, \qquad E(n_+, n_-) = \hbar\omega\,(n_+ + n_- + 1)\,. \tag{20}$$

The Fock states $|n_+, n_-\rangle$ indeed diagonalise the energy spectrum of the system.

1.5. The energy spectrum being given by

$$E(n_+, n_-) = \hbar\omega\,(n_+ + n_- + 1)\,, \qquad n_+, n_- = 0, 1, 2, \ldots, \tag{21}$$

is clearly degenerate in the values of $(n_+ + n_-)$. Namely, for a given value of $(n_+ + n_-) = N \geq 0$, there are as many states as there are partitions of the natural number N in the ordered sum of two natural numbers, namely $(N+1)$ states corresponding to the values $(n_+, n_-) = (N, 0), (N-1, 1), \cdots, (1, N-1), (0, N)$. Hence in a diagram of which the horizontal axis is $(n_+ - n_-)$ and the vertical one is $N = (n_+ + n_-)$, the spectrum of states is organised in a triangular shape which is symmetrical with respect to the vertical axis. The tip of the triangle, corresponding to the state $|n_+ = 0, n_- = 0\rangle$, is at the value $E(0,0) = \hbar\omega$, while starting from that ground state all other excitation states at each of the levels $N \geq 0$ are equally spaced in energy, with a gap $\Delta E = \hbar\omega$, and a degeneracy of $(N+1)$ states organised symmetrically with respect to the value $(n_+ - n_- = 0)$.

1.6. In three dimensional euclidean space the angular momentum is the vector quantity defined by $\vec{L} = m\vec{r} \times \dot{\vec{r}}$. When motion is restricted to a plane, only the component of $\vec{L}$ perpendicular to that plane is non vanishing, with the value $L = m(x_1\dot{x}_2 - x_2\dot{x}_1)$. In terms of the phase space coordinates we thus have

$$L = x_1 p_2 - x_2 p_1. \tag{22}$$

In other words, the angular momentum of the system is a phase space observable.

The corresponding quantum operator is thus,

$$\hat{L} = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1, \tag{23}$$

in which no ordering ambiguities arise. It now suffices to substitute in this expression the relations in (8) to find,

$$\hat{L} = -i\hbar\left(a_1^\dagger\, a_2 \;-\; a_2^\dagger\, a_1\right), \tag{24}$$

and in turn, when using (12),

$$\hat{L} = \hbar\left(a_+^\dagger\, a_+ \;-\; a_-^\dagger\, a_-\right). \tag{25}$$

Note that we have,

$$\left[\hat{L}, a_\pm\right] = \mp\,\hbar\, a_\pm, \qquad \left[\hat{L}, a_\pm^\dagger\right] = \pm\,\hbar\, a_\pm^\dagger, \tag{26}$$

showing that $a_+^\dagger$ and a_- carry both a unit $(+1)$ of angular momentum in units of $\hbar$, while $a_-^\dagger$ and a_+ a unit (-1).

Consequently the helicity Fock states $|n_+, n_-\rangle$ are precisely the eigenstates of the angular momentum operator, with

$$\hat{L}\,|n_+, n_-\rangle = \hbar\,(n_+ \;-\; n_-)\;|n_+, n_-\rangle. \tag{27}$$

Note thus that the spectrum of energy eigenvalues drawn in the diagram of the previous question is as function of the value of the angular momentum of energy eigenstates in units of $\hbar$.

1.7. In the Heisenberg picture the time dependence of the Fock algebra generators is governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}a_\pm(t) = \left[a_\pm(t),\hat{H}\right], \qquad i\hbar\frac{d}{dt}a^\dagger_\pm(t) = \left[a^\dagger_\pm(t),\hat{H}\right], \tag{28}$$

of which the solution is,

$$a_\pm(t) = e^{\frac{i}{\hbar}t\hat{H}}\,a_\pm\,e^{-\frac{i}{\hbar}t\hat{H}}, \qquad a^\dagger_\pm(t) = e^{\frac{i}{\hbar}t\hat{H}}\,a^\dagger_\pm(t)\,e^{-\frac{i}{\hbar}t\hat{H}}, \tag{29}$$

where $a_\pm$ and $a^\dagger_\pm$ are the values of these operators at $t = 0$ (at which the quantisation of the system is considered).

Since we simply have,

$$\left[a_\pm,\hat{H}\right] = \hbar\omega\,a_\pm, \qquad \left[a^\dagger_\pm,\hat{H}\right] = -\hbar\omega\,a^\dagger_\pm, \tag{30}$$

it follows that

$$a_\pm(t) = a_\pm\,e^{-i\omega t}, \qquad a^\dagger_\pm(t) = a^\dagger_\pm\,e^{i\omega t}. \tag{31}$$

These results agree of course with those which would follow from the basic identity

$$e^A\,B\,e^{-A} = B + [A,B] + \frac{1}{2!}\,[A,[A,B]] + \cdots, \tag{32}$$

with B standing for any of the operators $(a_\pm, a^\dagger_\pm)$, and A for $it\hat{H}/\hbar$.

Using then the relations in (13), it follows that the time dependence in the Heisenberg picture of the coordinate operators is,

$$\begin{aligned} \hat{x}_1(t) &= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left[(a_+ + a_-)\,e^{-i\omega t} + \left(a^\dagger_+ + a^\dagger_-\right)\,e^{i\omega t}\right], \\ \hat{x}_2(t) &= \frac{i}{2}\sqrt{\frac{\hbar}{m\omega}}\left[(a_+ - a_-)\,e^{-i\omega t} - \left(a^\dagger_+ - a^\dagger_-\right)\,e^{i\omega t}\right]. \end{aligned} \tag{33}$$

Introducing then the operators $\alpha_\pm = \sqrt{\hbar}\,a_\pm$ and $\alpha^\dagger_\pm = \sqrt{\hbar}\,a^\dagger_\pm$ which are such that $\left[\alpha_\pm, \alpha^\dagger\pm\right] = i\hbar\mathbb{I}$, the classical solutions for the coordinates are obtained in the limit $\hbar \to 0$, namely,

$$\begin{aligned} x_1(t) &= \frac{1}{2\sqrt{m\omega}}\left[(\alpha_+ + \alpha_-)\,e^{-i\omega t} + \left(\alpha^*_+ + \alpha^*_-\right)\,e^{i\omega t}\right], \\ x_2(t) &= \frac{i}{2\sqrt{m\omega}}\left[(\alpha_+ - \alpha_-)\,e^{-i\omega t} - \left(\alpha^*_+ - \alpha^*_-\right)\,e^{i\omega t}\right], \end{aligned} \tag{34}$$

$\alpha_\pm$ and $\alpha^*_\pm$ now corresponding to complex integration constants. Note that if one defines

$$\alpha_1 = \frac{1}{\sqrt{2}}\,(\alpha_+ + \alpha_-), \qquad \alpha_2 = \frac{i}{\sqrt{2}}\,(\alpha_+ - \alpha_-), \tag{35}$$

the classical solution also reads,

$$x_i(t) = \frac{1}{\sqrt{2m\omega}}\left(\alpha_i\,e^{-i\omega t} + \alpha^*_i\,e^{i\omega t}\right), \qquad i = 1, 2, \tag{36}$$

which of course is the usual solution for a single harmonic oscillator, for each independent cartesian coordinate.

Clearly it is a general feature that in the classical limit $\hbar \to 0$ the quantum solution to the Schrödinger equation in the Heisenberg picture goes over to the classical solution for the time dependence of the classical trajectories of the system, whether in configuration space or phase space.

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2. **The Generalised Landau Problem**[2]

The present Problem considers a nonrelativistic charged particle, of mass m and charge q, whose motion is restricted to a plane. Relative to an inertial frame of which the basis of orthonormalised vectors is $(\hat{x}_1, \hat{x}_2)$ lying inside that plane, the position vector of the particle, $\vec{r}$, possesses components (x_1, x_2). The origin of that frame is chosen to coincide with the symmetry center of a spherically symmetric harmonic force acting on the particle, of which the potential energy is $\frac{1}{2}k\vec{r}^{\,2}$, $k \geq 0$. The particle is also subjected to a constant and homogeneous magnetic field, $\vec{B}_0 = B_0 \hat{x}_1 \times \hat{x}_2$, which is perpendicular to the plane, as well as a constant and homogeneous electric field $\vec{E}_0$ lying inside the plane, with components (E_{01}, E_{02}) relative to the basis $(\hat{x}_1, \hat{x}_2)$. As it turns out quite many of the results of Problem 1. of this Assignment as well as of Problem 1. of Assignment 1. are of use in solving the present Problem.

2.1. In order to take to the most advantage of the gauge freedom in choosing the electromagnetic scalar and vector potentials when expressing the action for the system, let us first identify all possible time independent potentials which produce this configuration of electric and magnetic fields. In the notations of the Course, from past experience we know already that the following choice applies,

$$\Phi(\vec{r}\,) = -\vec{r}\cdot\vec{E}_0, \qquad \vec{A}(\vec{r}\,) = -\frac{1}{2}B_0 x_2\,\hat{x}_1 \;+\; \frac{1}{2}B_0 x_1\,\hat{x}_2.$$

Explain then why the most general class of these electromagnetic potentials which is time independent is given as

$$\Phi'(\vec{r}\,) = -\vec{r}\cdot\vec{E}_0 \,+\, \Phi_0, \qquad \vec{A}'(\vec{r}\,) = \left(-\frac{1}{2}B_0\,x_2 \,+\, \partial_1\chi_0(\vec{r}\,)\right)\hat{x}_1 \;+\; \left(\frac{1}{2}B_0\,x_1 \,+\, \partial_2\chi_0(\vec{r}\,)\right)\hat{x}_2,$$

where Φ_0 is an arbitrary constant and $\chi_0(\vec{r}\,)$ an arbitrary function of space, each having the appropriate physical dimensions.

2.2. Indicate then why the Lagrange function of this system may be chosen to be given as,

$$L = \frac{1}{2}m\dot{x}_i^2 \,+\, x_i E_i \,-\, \frac{1}{2}B\epsilon_{ij}\,\dot{x}_i x_j \,-\, \frac{1}{2}kx_i^2 \,+\, \dot{x}_i\,\partial_i\chi \,-\, q\Phi_0,$$

where we have redefined $\vec{E} = q\vec{E}_0$, $B = qB_0$ and $\chi(\vec{r}\,) = q\chi_0(\vec{r}\,)$. Note that by an appropriate choice of the orientation of the basis $(\hat{x}_1, \hat{x}_2)$ in the plane, without loss of generality we may always assume that $B > 0$, which shall be done all throughout.

As always with actions that are second order forms in the degrees of freedom, it is useful to bring the total potential energy of the system in a purely quadratic diagonal form, in the present case,

$$V(\vec{r}\,) = \frac{1}{2}k\vec{r}^{\,2} \,-\, \vec{r}\cdot\vec{E} = \frac{1}{2}k\vec{u}^{\,2} - \frac{1}{2}k\vec{b}^{\,2}.$$

Show that for the present system the relevant change of variable is

$$u_i = x_i \,-\, b_i, \qquad \dot{u}_i = \dot{x}_i, \qquad b_i = \frac{1}{k}E_i.$$

Explain then how the following gauge choice for the electromagnetic potentials,

$$\chi(\vec{r}\,) = \frac{1}{2}B\epsilon_{ij}\,x_i b_j \,+\, \bar{\chi}, \qquad q\Phi_0 = \frac{1}{2}k\vec{b}^{\,2},$$

where $\bar{\chi}$ is an arbitrary constant, allows one to finally bring the Lagrangian of the system in the following simple form,

$$L = \frac{1}{2}m\dot{u}_i^2 \,-\, \frac{1}{2}ku_i^2 \,-\, \frac{1}{2}B\epsilon_{ij}\,\dot{u}_i u_j.$$

2.3. The Hamiltonian formulation of the system then readily follows from the above Lagrangian. The momenta canonically conjugate to u_i are denoted π_i, with the Poisson brackets $\{u_i, \pi_j\} = \delta_{ij}$. Explain why the canonical Hamiltonian of the system is then given as,

$$H = \frac{1}{2m}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}u_j\right)^2 \,+\, \frac{1}{2}ku_i^2 = \frac{1}{2m}\pi_i^2 \,+\, \frac{1}{2}m\omega^2 u_i^2 \,-\, \frac{1}{2}\omega_c\,\epsilon_{ij}u_i\pi_j,$$

[2] Further and relevant discussion of this system may be found in, Jan Govaerts, M. Norbert Hounkonnou and Habatwa V. Mweene, *J. Phys. A: Math. Theor.* **42** (2009) 485209 (19pp), [e-print: `arXiv:0909.2659 [hep-th]`].

where

$$\omega = \sqrt{\frac{k}{m} + \frac{1}{4}\omega_c^2} > 0, \qquad \omega_c = \frac{B}{m} > 0.$$

In its last form, this expression for the Hamiltonian is indeed very suggestive, in view of the results established in Problem 1. of this Assignment.

2.4. Canonical quantisation of the system then proceeds straightforwardly using the results of Problem 1. By introducing the relevant Fock operators $(a_i, a_i^\dagger)$ related this time to the operators $(\hat{u}_i, \hat{\pi}_i)$, and next the corresponding helicity ones $(a_\pm, a_\pm^\dagger)$, show that the quantum Hamiltonian operator of the system reduces to

$$\hat{H} = \hbar\omega_-\, a_+^\dagger a_+ \,+\, \hbar\omega_+\, a_-^\dagger a_- \,+\, \hbar\omega, \qquad \omega_\pm = \omega \pm \frac{1}{2}\omega_c.$$

2.5. Identify then explicitly the energy eigenspectrum of the system, namely the energy eigenvalues and eigenstates. Draw in a diagram of which the horizontal axis is the variable $(n_+ - n_-)$ and the vertical one the energy value, the spectrum of energy states, where n_+ (resp., n_-) is the eigenvalue of the number operator $a_+^\dagger a_+$ (resp., $a_-^\dagger a_-$).

2.6. In the Heisenberg picture, solve for the time dependence of the operators $a_\pm$ and $a_\pm^\dagger$ and identify the time dependence of the position observables $\hat{x}_i(t)$ $(i = 1, 1)$. Infer then what the solutions to the dynamics are for the classical coordinates $x_i(t)$ $(i = 1, 2)$.

2.7. Consider now the system in the limit where, first, the electric field is taken away $(\vec{E} = \vec{0})$, and next, the harmonic potential $(k = 0)$. How does the energy spectrum then look like? Describe the energy degeneracies that you observe (these are known as "the Landau levels" of the Landau problem). Show that in this limit, in the Heisenberg picture, the position operators are now of the form $\hat{x}_i(t) = \hat{x}_i^c + \hat{x}_i^{\rm circ}(t)$, with the following magnetic centre coordinates,

$$\hat{x}_i^c = \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ \,+\, a_+^\dagger\right), \qquad \hat{x}_2^c = i\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ \,-\, a_+^\dagger\right),$$

and the following coordinates of the circular trajectory (about the static magnetic centre),

$$\hat{x}_1^{\rm circ}(t) = \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\, e^{-i\omega_c t} \,+\, a_-^\dagger\, e^{i\omega_c t}\right), \qquad \hat{x}_2^{\rm circ}(t) = -i\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\, e^{-i\omega_c t} \,-\, a_-^\dagger\, e^{i\omega_c t}\right).$$

What are the commutation relations of these different coordinates? In particular note how the magnetic center coordinates define a noncommutative geometry in the euclidean plane.

2.8. There must exist an underlying explanation for the infinite degeneracies of the Landau levels in the absence of both the electric field and the harmonic potential. Since this degeneracy is clearly related to magnetic center sector of the system which may be positioned anywhere inside the plane, presumably it is translation invariance of the Landau problem which is the symmetry which accounts for this degeneracy. Indeed, the Lagrange function of the system, which now reads

$$L = \frac{1}{2}m\dot{x}_i^2 \,-\, \frac{1}{2}B\epsilon_{ij}\,\dot{x}_i x_j,$$

is then invariant up to a surface term under constant translations in the plane, $\vec{r}\,' = \vec{r} + \vec{a}$, $\vec{a}$ being an arbitrary constant vector. Explain why the corresponding conserved Noether charge is given as $P_i = \pi_i \,-\, \frac{1}{2}B\,\epsilon_{ij}x_j$, π_i being the momenta conjugate to the coordinates x_i.

As operators, show that these generators of translations in the plane indeed commute with $\hat{H}$, hence define a symmetry of the quantum system, of which the algebra is (this algebra is commutative when $B = 0$),

$$\left[\hat{P}_i, \hat{P}_j\right] = -i\hbar\, B\, \epsilon_{ij}\, \mathbb{I}.$$

Establish that as a matter of fact one has $\hat{P}_i = -B\epsilon_{ij}\,\hat{x}_j^c$, showing that up to normalisation, the magnetic centre coordinates are indeed the generators of translations in the plane, are conjugate to one another, which is why they define a noncommutative geometry in the plane, and map all quantum states belonging to a same Landau level into one another hence explaining the existence of the infinite degeneracies of these energy levels.

2.1. It should indeed be quite obvious that the following configuration of electromagnetic potentials,

$$\Phi(\vec{r}) = -\vec{r}\cdot\vec{E}_0, \qquad \vec{A}(\vec{r}) = -\frac{1}{2}B_0 x_2\,\hat{x}_1 \;+\; \frac{1}{2}B_0 x_1\,\hat{x}_2, \tag{1}$$

reproduces the described configuration of electric and magnetic fields, $\vec{E}_0$ and $\vec{B}_0$, through the general gauge invariant relations

$$\vec{E}(t,\vec{r}) = -\vec{\nabla}\Phi(t,\vec{r}) \,-\, \partial_t\vec{A}(t,\vec{r}), \qquad \vec{B}(t,\vec{r}) = \vec{\nabla}\times\vec{A}(t,\vec{r}). \tag{2}$$

However gauge transformations of the potentials,

$$\Phi'(t,\vec{r}) = \Phi(t,\vec{r}) - \partial_t\chi(t,\vec{r}), \qquad \vec{A}'(t,\vec{r}) = \vec{A}(t,\vec{r}) + \vec{\nabla}\chi(t,\vec{r}), \tag{3}$$

with $\chi(t,\vec{r})$ being an arbitrary time and space dependent function, parametrise the complete set of potentials associated to a given configuration of electric and magnetic fields. In the present case by requiring that potentials remain time independent leads to the following conditions,

$$\partial_t^2\chi(t,\vec{r}) = 0, \qquad \partial_t\vec{\nabla}\chi(t,\vec{r}) = \vec{0}, \tag{4}$$

of which the complete general solution is simply (see Solution to Problem 1. of Assignment 1.),

$$\chi(t,\vec{r}) = -t\,\Phi_0 \,+\, \chi_0(\vec{r}), \tag{5}$$

where Φ_0 is an arbitrary constant having the same physical dimension as Φ, and $\chi_0(\vec{r})$ being an arbitrary function of space with its own appropriate physical dimensions. Consequently, the general class of time independent electromagnetic potentials associated to the considered configuration of electric and magnetic fields is,

$$\Phi'(\vec{r}) = -\vec{r}\cdot\vec{E}_0 \,+\, \Phi_0, \qquad \vec{A}'(\vec{r}) = \left(-\frac{1}{2}B_0\,x_2 \,+\, \partial_1\chi_0(\vec{r})\right)\hat{x}_1 \;+\; \left(\frac{1}{2}B_0\,x_1 \,+\, \partial_2\chi_0(\vec{r})\right)\hat{x}_2. \tag{6}$$

2.2. Given the general Lagrange function describing the coupling of a nonrelativistic massive charged particle to a background electromagnetic field, and the specific configuration of fields as constructed above for the electromagnetic potentials, it is clear that in the present case the Lagrange function is given as,

$$\begin{aligned} L &= \frac{1}{2}m\dot{\vec{r}}^{\,2} - q\Phi'(\vec{r}) + q\dot{\vec{r}}\cdot\vec{A}'(\vec{r}) - \frac{1}{2}k\vec{r}^{\,2} \\ &= \frac{1}{2}m\dot{x}_i^2 + x_iE_i - \frac{1}{2}B\epsilon_{ij}\dot{x}_ix_j - \frac{1}{2}kx_i^2 + \dot{x}_i\partial_i\chi(\vec{r}) - q\Phi_0, \end{aligned} \tag{7}$$

where $\vec{E} = q\vec{E}_0$, $B = qB_0 > 0$ and $\chi(\vec{r}) = q\chi_0(\vec{r})$. Considering then the total contribution of second order in the degrees of freedom x_i, it may readily be brought into a purely quadratic and diagonal form,

$$V(\vec{r}) = \frac{1}{2}kx_i^2 - x_iE_i = \frac{1}{2}k\left(x_i - \frac{1}{k}E_i\right)^2 - \frac{1}{2}k\,\frac{E_i^2}{k^2}, \tag{8}$$

namely,

$$V(\vec{r}) = \frac{1}{2}ku_i^2 - \frac{1}{2}kb_i^2, \qquad u_i = x_i \,-\, b_i, \qquad \dot{u}_i = \dot{x}_i, \qquad b_i = \frac{1}{k}E_i. \tag{9}$$

A substitution into the Lagrange function then finds,

$$L = \frac{1}{2}m\dot{u}_i^2 - \frac{1}{2}ku_i^2 - \frac{1}{2}B\epsilon_{ij}\dot{u}_i\,(u_j + b_j) + \dot{u}_i\partial_i\chi(\vec{r}) - q\Phi_0 + \frac{1}{2}kb_i^2. \tag{10}$$

It is now possible to adjust the choice of gauge potentials so as to cancel some terms, and obtain a most convenient form for the action of the system, by having

$$\partial_i\chi(\vec{r}) - \frac{1}{2}B\epsilon_{ij}b_j = 0, \qquad q\Phi_0 - \frac{1}{2}kb_i^2 = 0, \tag{11}$$

namely,

$$q\Phi_0 = \frac{1}{2}k\vec{b}^2, \qquad \chi(\vec{r}) = \frac{1}{2}B\epsilon_{ij}x_i b_j + \bar{\chi}, \tag{12}$$

with $\bar{\chi}$ being an arbitrary constant having the same physical dimension as $\chi(\vec{r}) = q\chi_0(\vec{r})$. With such a choice of gauge potentials, the Lagrange function of the system is finally given by the simple expression,

$$L = \frac{1}{2}m\dot{u}_i^2 - \frac{1}{2}ku_i^2 - \frac{1}{2}B\epsilon_{ij}\dot{u}_i u_j. \tag{13}$$

2.3. Given the form of the Lagrangian in (13), setting up the Hamiltonian formulation of the system is now straightforward. The canonically conjugate phase space variables are (u_i, π_i) with

$$\pi_i = \frac{\partial L}{\partial \dot{u}_i} = m\dot{u}_i - \frac{1}{2}B\epsilon_{ij}u_j, \qquad \dot{u}_i = \frac{1}{m}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}u_j\right), \qquad \{u_i, \pi_j\} = \delta_{ij}, \tag{14}$$

while the canonical Hamiltonian is,

$$\begin{aligned} H &= \dot{u}_i\pi_i - L = \frac{1}{m}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}u_j\right)\pi_i - \frac{1}{2m}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}u_j\right)^2 + \frac{1}{2}ku_i^2 + \frac{1}{2}\frac{B}{m}\epsilon_{ij}\left(\pi_i + \frac{1}{2}B\epsilon_{ik}u_k\right)u_j \\ &= \frac{1}{2m}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}u_j\right)^2 + \frac{1}{2}ku_i^2. \end{aligned} \tag{15}$$

By expanding this expression one finds,

$$\begin{aligned} H &= \frac{1}{2m}\pi_i^2 + \frac{1}{2}\left(k + \frac{1}{4}\frac{B^2}{m}\right)u_i^2 - \frac{1}{2}\frac{B}{m}\epsilon_{ij}u_i\pi_j \\ &= \frac{1}{2m}\pi_i^2 + \frac{1}{2}m\omega^2 u_i^2 - \frac{1}{2}\omega_c\epsilon_{ij}u_i\pi_j, \end{aligned} \tag{16}$$

where

$$\omega_c = \frac{B}{m} > 0, \qquad \omega = \sqrt{\frac{k}{m} + \frac{1}{4}\omega_c^2} > 0. \tag{17}$$

In this form, one recognizes the Hamiltonian of a spherically symmetric harmonic oscillator in the plane of angular frequency ω, to which is added a term proportional to the angular momentum of that oscillator, $\epsilon_{ij}u_i\pi_j$. Precisely this system has been solved in Problem 1., where it was shown that for the quantised system, both these observables are diagonalised together in the helicity basis of Fock states. Given this remark, the quantisation and resolution of this generalised Landau problem is immediate.

2.4. The quantised system is thus defined by the following properties for the relevant phase space operators,

$$[\hat{u}_i, \hat{\pi}_j] = i\hbar\delta_{ij}, \qquad \hat{H} = \frac{1}{2m}\hat{\pi}_i^2 + \frac{1}{2}m\omega^2\hat{u}_i^2 - \frac{1}{2}\omega_c\epsilon_{ij}\hat{u}_i\hat{\pi}_j. \tag{18}$$

Given the solution to Problem 1., let us first introduce the following Fock operators,

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{u}_i + \frac{i}{m\omega}\hat{\pi}_i\right), \qquad a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{u}_i - \frac{i}{m\omega}\hat{\pi}_i\right), \qquad \left[a_i, a_j^\dagger\right] = \mathbb{I}, \qquad i = 1, 2. \tag{19}$$

Next consider the helicity Fock algebra,

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp ia_2\right), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm ia_2^\dagger\right), \qquad \left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}. \tag{20}$$

In particular, inverting these relations one has

$$\hat{u}_1 = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ + a_- + a_+^\dagger + a_-^\dagger\right) \quad , \qquad \hat{u}_2 = \frac{i}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ - a_- - a_+^\dagger + a_-^\dagger\right),$$
$$\hat{\pi}_1 = -i\frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ + a_- - a_+^\dagger - a_-^\dagger\right) \quad , \qquad \hat{\pi}_2 = \frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ - a_- + a_+^\dagger - a_-^\dagger\right). \tag{21}$$

From the discussion in Problem 1., we know that we then have,

$$\frac{1}{2m}\hat{\pi}_i^2 + \frac{1}{2}m\omega^2\hat{u}_i^2 = \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right), \qquad \epsilon_{ij}\hat{u}_i\pi_j = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \tag{22}$$

so that finally the Hamiltonian operator of the generalised Landau problem reads,

$$\begin{aligned}\hat{H} &= \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right) - \frac{1}{2}\hbar\omega_c\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) \\ &= \hbar\omega_-\, a_+^\dagger a_+ + \hbar\omega_+\, a_-^\dagger a_- + \hbar\omega,\end{aligned} \tag{23}$$

with

$$\omega_+ = \omega + \frac{1}{2}\omega_c, \qquad \omega_- = \omega - \frac{1}{2}\omega_c. \tag{24}$$

2.5. It is clear that the Hamiltonian operator is diagonalised by the orthonormalised helicity Fock states defined by

$$|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!\ n_-!}}\left(a_+^\dagger\right)^{n_+}\left(a_-^\dagger\right)^{n_-}|0,0\rangle, \qquad n_+, n_- = 0, 1, 2, \ldots, \tag{25}$$

$|0,0\rangle$ being the Fock vacuum of the system, such that $a_\pm|0,0\rangle = 0$ and $\langle 0,0|0,0\rangle = 1$, these states being such that,

$$\langle n_+, n_-|m_+, m_-\rangle = \delta_{n_+,m_+}\delta_{n_-,m_-}. \tag{26}$$

Indeed, since $a_\pm^\dagger a_\pm|n_+,n_-\rangle = n_\pm|n_+,n_-\rangle$, we simply have,

$$\hat{H}|n_+,n_-\rangle = E(n_+,n_-)\,|n_+,n_-\rangle, \qquad E(n_+,n_-) = \hbar\omega_-\,n_+ \;+\; \hbar\omega_+\,n_- \;+\; \hbar\omega. \tag{27}$$

In a diagram in which these energy values are displayed as a function of the angular momentum in units of $\hbar$, namely the values $(n_+ - n_-)$, for any fixed value of $n_- = 0, 1, 2, \cdots$ the energy values lie along a straight line with slope $\hbar\omega_-$ and starting at the point of angular momentum $(-n_-)$ with energy $\hbar(\omega_+ n_- + \omega)$. In other words, the spectrum of the spherically symmetric harmonic oscillator of Problem 1., which is symmetric with respect to the energy axis, is now tilted towards the right of the diagram.

2.6. Given the considerations already detailed in the solution to Problem 1., it should be clear that since we have the commutation relations,

$$\left[a_\pm, \hat{H}\right] = \hbar\omega_\mp\, a_\pm, \qquad \left[a_\pm^\dagger, \hat{H}\right] = -\hbar\omega_\mp\, a_\pm^\dagger, \tag{28}$$

in the Heisenberg picture the time dependence of the Fock operators is given as,

$$a_\pm(t) = a_\pm\, e^{-i\omega_\mp t}, \qquad a_\pm^\dagger(t) = a_\pm^\dagger\, e^{i\omega_\mp t}, \tag{29}$$

$(a_\pm, a_\pm^\dagger)$ being the helicity Fock operators as defined through canonical quantisation at time $t = 0$.

In view of the relations in (21) as well as the definition $u_i = x_i - b_i$, it follows that in the Heisenberg picture the time dependence of the position observables is,

$$\begin{aligned}\hat{x}_1(t) &= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+\, e^{-i\omega_- t} + a_-\, e^{-i\omega_+ t} + a_+^\dagger\, e^{i\omega_- t} + a_-^\dagger\, e^{i\omega_+ t}\right) + b_1\,\mathbb{I} \\ \hat{x}_2(t) &= \frac{i}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+\, e^{-i\omega_- t} - a_-\, e^{-i\omega_+ t} - a_+^\dagger\, e^{i\omega_- t} + a_-^\dagger\, e^{i\omega_+ t}\right) + b_2\,\mathbb{I}.\end{aligned} \tag{30}$$

By absorbing Planck's constant $\hbar$ into the normalisation of the Fock operators, $\alpha_\pm = \sqrt{\hbar}\, a_\pm$ and $\alpha_\pm^\dagger = \sqrt{\hbar}\, a_\pm^\dagger$, with $\left[\alpha_\pm, \alpha_\pm^\dagger\right] = \mathbb{I}$, in the classical limit $\hbar \to 0$ one has the following representation of the classical trajectories of the system,

$$\begin{aligned} x_1(t) &= \frac{1}{2\sqrt{m\omega}}\left(\alpha_+\, e^{-i\omega_-\, t} + \alpha_-\, e^{-i\omega_+\, t} + \alpha_+^*\, e^{i\omega_-\, t} + \alpha_-^*\, e^{i\omega_+\, t}\right) + b_1 \\ x_2(t) &= \frac{i}{2\sqrt{m\omega}}\left(\alpha_+\, e^{-i\omega_-\, t} - \alpha_-\, e^{-i\omega_+\, t} - \alpha_+^*\, e^{i\omega_-\, t} + \alpha_-^*\, e^{i\omega_+\, t}\right) + b_2, \end{aligned} \tag{31}$$

$\alpha_\pm$ and $\alpha_\pm^*$ then corresponding to complex integration constants.

2.7. By first setting $\vec{E} = \vec{0}$, and next $k = 0$, one finds,

$$b_i = 0, \qquad \omega = \frac{1}{2}\omega_c, \qquad \omega_+ = \omega_c, \qquad \omega_- = 0. \tag{32}$$

Consequently,

$$\hat{H} = \hbar\omega_c\left(a_-^\dagger a_- + \frac{1}{2}\right), \tag{33}$$

so that the energy eigenstates are still the helicity Fock states $|n_+, n_-\rangle$ with the eigenvalues

$$E(n_+, n_-) = \hbar\omega_c\left(n_- + \frac{1}{2}\right), \qquad \hat{H}\,|n_+, n_-\rangle = E(n_+, n_-)\,|n_+, n_-\rangle. \tag{34}$$

The diagram representing this spectrum is now organised into horizontal lines at the values $\hbar\omega_c(n_- + 1/2)$ for any fixed value of $n_- = 0, 1, 2, \cdots$, and starting towards the right from the angular momentum value $(-n_-)$. In other words, the diagram of the spherically symmetric harmonic oscillator has now completely tilted over, to lead to energy levels which are infinitely degenerate in the values $n_+ = 0, 1, 2, \cdots$ for any fixed value of $n_- = 0, 1, 2, \cdots$. Landau levels thus correspond to the subspaces of the full Hilbert space of the system which are spanned by the states $\{|n_+, n_-\rangle, n_+ = 0, 1, 2, \cdots\}$, for any fixed value of $n_- = 0, 1, 2, \cdots$.

Considering then in the Heisenberg picture the time dependence of the plane coordinates in (30), when only the magnetic field coupling is still present these expressions reduce to,

$$\hat{x}_i(t) = \hat{x}_i^c + \hat{x}_i^{\rm circ}(t), \qquad i = 1, 2, \tag{35}$$

with the magnetic centre coordinate operators

$$\hat{x}_1^c = \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ + a_+^\dagger\right), \qquad \hat{x}_2^c = i\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ - a_+^\dagger\right), \tag{36}$$

while the circular component of the motion pinned at the magnetic centre position is described by,

$$\hat{x}_1^{\rm circ}(t) = \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\, e^{-i\omega_c\, t} + a_-^\dagger\, e^{i\omega_c\, t}\right), \qquad \hat{x}_2^{\rm circ}(t) = -i\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\, e^{-i\omega_c\, t} - a_-^\dagger\, e^{i\omega_c t}\right). \tag{37}$$

Quite obviously, the coordinates $\hat{x}_i^c$ and $\hat{x}_i^{\rm circ}(t)$ commute with one another, but among themselves we have,

$$[\hat{x}_1^c, \hat{x}_2^c] = -\frac{i\hbar}{B}, \qquad [\hat{x}_1^{\rm circ}(t), \hat{x}_2^{\rm circ}(t)] = \frac{i\hbar}{B}, \tag{38}$$

while of course we still have $[\hat{x}_i(t), \hat{x}_j(t)] = 0$. Hence the magnetic centre coordinates define a noncommutative geometry associated to the euclidean plane. When restricted to any of the Landau levels, that subspace of Hilbert space defines the representation space of the noncommutative euclidean plane.

2.8. In the absence of both the electric field and the harmonic potential, the Lagrange function of the Landau problem reduces to,

$$L = \frac{1}{2} m \dot{x}_i^2 - \frac{1}{2} B \epsilon_{ij} \dot{x}_i x_j. \tag{39}$$

Under a constant translation in the plane of translation vector $\vec{a}$, $\vec{r}\,' = \vec{r} + \vec{a}$, it should be clear that the Lagrange function transforms according to,

$$L' = L - \frac{1}{2} B \epsilon_{ij} \dot{x}_i a_j = L + \frac{d}{dt}\left(-\frac{1}{2} B \epsilon_{ij} x_i a_j\right). \tag{40}$$

Given the general discussion of Noether's (first) theorem discussed in the Course, and in particular the expression for Noether charges, it follows that for the present transformation the two Noether charges associated to the two independent translations in the plane of the Landau problem are, both classical and thus quantum mechanically,

$$P_i = \pi_i - \frac{1}{2} B \epsilon_{ij} x_j, \qquad \hat{P}_i = \hat{\pi}_i - \frac{1}{2} B \epsilon_{ij} \hat{x}_j, \tag{41}$$

π_i being the momentum conjugate to the coordinate x_i (in agreement with the previous notations for this Problem since we now have $u_i = x_i$).

Note that the Hamiltonian operator now reads,

$$\hat{H} = \frac{1}{2m}\left(\hat{\pi}_i + \frac{1}{2} B \epsilon_{ij} \hat{x}_j\right)^2. \tag{42}$$

On the other hand a direct calculation finds,

$$\left[\hat{P}_i, \hat{\pi}_j + \frac{1}{2} B \epsilon_{j\ell} \hat{x}_\ell\right] = \left[\hat{\pi}_i - \frac{1}{2} B \epsilon_{ik} \hat{x}_k, \hat{\pi}_j + \frac{1}{2} B \epsilon_{j\ell} \hat{x}_\ell\right] = \frac{1}{2} i\hbar B \epsilon_{ij} - \frac{1}{2} i\hbar B \epsilon_{ij} = 0. \tag{43}$$

Consequently,

$$\left[\hat{P}_i, \hat{H}\right] = 0, \tag{44}$$

showing that the observables $\hat{P}_i$ indeed generate symmetries of the system, and define conserved quantities corresponding in fact to the total momentum of the system (namely the velocity momentum of the particle to which must be added a contribution due to the magnetic field coupling, $P_i = m\dot{x}_i - B\epsilon_{ij} x_j$. After all the equations of motion of the system are, $m\ddot{x}_i - B\epsilon_{ij}\dot{x}_j = 0$, namely $\dot{P}_i = 0$).

Furthermore a direct calculation finds

$$\left[\hat{P}_1, \hat{P}_2\right] = \left[\hat{\pi}_1 - \frac{1}{2} B \hat{x}_2, \hat{\pi}_2 + \frac{1}{2} B \hat{x}_1\right] = -i\hbar B, \qquad \left[\hat{P}_i, \hat{P}_j\right] = -i\hbar B\, \epsilon_{ij}. \tag{45}$$

Note that usually generators of translations commute (as they do in the present system in the absence of the magnetic field, in which case the particle is free), but not in the present instance because of the magnetic coupling.

Finally, when using the relations in (21) a direct substitution finds,

$$\hat{P}_1 = \hat{\pi}_1 - \frac{1}{2} B \hat{x}_2 = -im\omega_c \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ - a_+^\dagger\right) = -B\, \hat{x}_2^c, \quad \hat{P}_2 = \hat{\pi}_2 + \frac{1}{2} B \hat{x}_1 = m\omega_c \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ + a_+^\dagger\right) = B\, \hat{x}_1^c \tag{46}$$

namely,

$$\hat{P}_i = -B\, \epsilon_{ij}\, \hat{x}_j^c. \tag{47}$$

In other words, up to a normalisation factor given by the magnetic field, the magnetic centre coordinates are the generators of translations in the plane, while clearly being constructed out of the helicity Fock generators a_+ and $a_+^\dagger$ only, they transform states of any given Landau level among themselves only. Consequently, it is the symmetry of the Landau problem under translations in the plane which explains the infinite degeneracy of the energy levels of the Landau problem. Furthermore these magnetic centre coordinates are conjugate to one another and thus do not commute, but rather define a noncommutative geometry associated to the euclidean plane.
