An Introduction to Quantum Physics
and
Relativistic Quantum Field Theory

AIMS Lectures: 24 January – 11 February 2011
Assignment 2:
To be handed-in by Wednesday 9 February 2011, 23:00 PM

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1. **The Two Dimensional Spherically Symmetric Quantum Harmonic Oscillator**
Consider the Lagrange function for the two dimensional spherically symmetric harmonic oscillator of mass m and angular frequency ω in the euclidean plane of cartesian coordinates $x_i(t)$ $(i=1,2)$,

$$L=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}m\omega^2\left(x_1^2+x_2^2\right).$$

1.1. Establish the Hamiltonian formulation of this system in terms of its canonical phase space coordinates $(x_1,p_1;x_2,p_2)$.

1.2. Apply the rules of canonical quantisation in order to define the quantum Hamiltonian operator $\hat{H}$ and the commutation relations of the basic operators $\hat{x}_i$ and $\hat{p}_i$ at time $t=0$.

1.3. Introduce the operators

$$a_i=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i+\frac{i}{m\omega}\hat{p}_i\right),\qquad a_i^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i-\frac{i}{m\omega}\hat{p}_i\right),\qquad i=1,2,$$

and next

$$a_\pm=\frac{1}{\sqrt{2}}\left(a_1\mp ia_2\right),\qquad a_\pm^\dagger=\frac{1}{\sqrt{2}}\left(a_1^\dagger\pm ia_2^\dagger\right).$$

Show that each of the sets of operators $(a_i,a_i^\dagger)$ and $(a_\pm,a_\pm^\dagger)$ define two commuting Fock algebras,

$$\left[a_i,a_j^\dagger\right]=\delta_{ij}\mathbb{I},\qquad \left[a_\pm,a_\pm^\dagger\right]=\mathbb{I}.$$

1.4. Express the quantum Hamiltonian $\hat{H}$ in terms of the annihilation and creation operators $a_\pm$ and $a_\pm^\dagger$, and show how its eigenspectrum may then be constructed in terms of the Fock states associated to the Fock algebra $(a_\pm,a_\pm^\dagger)$.

1.5. Draw in a diagram of which the horizontal axis is the variable (n_+-n_-) and the vertical one the energy value, the spectrum of energy states, where n_+ (resp., n_-) is the eigenvalue of the number operator $a_+^\dagger a_+$ (resp., $a_-^\dagger a_-$). Describe the degeneracies that exist for this spectrum.

1.6. Consider the angular momentum of the system, $L=m\left(x_1\dot{x}_2-x_2\dot{x}_1\right)$, and express it as a function of the phase space variables. Identify the corresponding quantum operator, and find its expression in terms of the Fock operators $(a_\pm,a_\pm^\dagger)$. Determine the eigenspectrum of the quantum angular momentum operator $\hat{L}$, namely the spectrum of its eigenvalues and eigenstates.

1.7. In the Heisenberg picture, solve for the time dependence of the operators $a_\pm$ and $a_\pm^\dagger$ and identify the time dependence of the position observables $\hat{x}_i(t)$ $(i=1,2)$. Infer then what the solutions to the dynamics are for the classical coordinates $x_i(t)$ $(i=1,2)$.

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2. **The Generalised Landau Problem**[1]

The present Problem considers a nonrelativistic charged particle, of mass m and charge q, whose motion is restricted to a plane. Relative to an inertial frame of which the basis of orthonormalised vectors is $(\hat{x}_1, \hat{x}_2)$ lying inside that plane, the position vector of the particle, $\vec{r}$, possesses components (x_1, x_2). The origin of that frame is chosen to coincide with the symmetry center of a spherically symmetric harmonic force acting on the particle, of which the potential energy is $\frac{1}{2}k\vec{r}^{\,2}$, $k \geq 0$. The particle is also subjected to a constant and homogeneous magnetic field, $\vec{B}_0 = B_0\hat{x}_1 \times \hat{x}_2$, which is perpendicular to the plane, as well as a constant and homogeneous electric field $\vec{E}_0$ lying inside the plane, with components (E_{01}, E_{02}) relative to the basis $(\hat{x}_1, \hat{x}_2)$. As it turns out quite many of the results of Problem 1. of this Assignment as well as of Problem 1. of Assignment 1. are of use in solving the present Problem.

2.1. In order to take to the most advantage of the gauge freedom in choosing the electromagnetic scalar and vector potentials when expressing the action for the system, let us first identify all possible time independent potentials which produce this configuration of electric and magnetic fields. In the notations of the Course, from past experience we know already that the following choice applies,

$$\Phi(\vec{r}) = -\vec{r}\cdot\vec{E}_0, \qquad \vec{A}(\vec{r}) = -\frac{1}{2}B_0 x_2\,\hat{x}_1 \;+\; \frac{1}{2}B_0 x_1\,\hat{x}_2.$$

Explain then why the most general class of these electromagnetic potentials which is time independent is given as

$$\Phi'(\vec{r}) = -\vec{r}\cdot\vec{E}_0 \;+\; \Phi_0, \qquad \vec{A}'(\vec{r}) = \left(-\frac{1}{2}B_0\,x_2 \;+\; \partial_1\chi_0(\vec{r})\right)\hat{x}_1 \;+\; \left(\frac{1}{2}B_0\,x_1 \;+\; \partial_2\chi_0(\vec{r})\right)\hat{x}_2,$$

where Φ_0 is an arbitrary constant and $\chi_0(\vec{r})$ an arbitrary function of space, each having the appropriate physical dimensions.

2.2. Indicate then why the Lagrange function of this system may be chosen to be given as,

$$L = \frac{1}{2}m\dot{x}_i^2 \;+\; x_i E_i \;-\; \frac{1}{2}B\epsilon_{ij}\,\dot{x}_i x_j \;-\; \frac{1}{2}kx_i^2 \;+\; \dot{x}_i\,\partial_i\chi \;-\; q\Phi_0,$$

where we have redefined $\vec{E} = q\vec{E}_0$, $B = qB_0$ and $\chi(\vec{r}) = q\chi_0(\vec{r})$. Note that by an appropriate choice of the orientation of the basis $(\hat{x}_1, \hat{x}_2)$ in the plane, without loss of generality we may always assume that $B > 0$, which shall be done all throughout.

As always with actions that are second order forms in the degrees of freedom, it is useful to bring the total potential energy of the system in a purely quadratic diagonal form, in the present case,

$$V(\vec{r}) = \frac{1}{2}k\vec{r}^{\,2} \;-\; \vec{r}\cdot\vec{E} = \frac{1}{2}k\vec{u}^{\,2} - \frac{1}{2}k\vec{b}^{\,2}.$$

Show that for the present system the relevant change of variable is

$$u_i = x_i \;-\; b_i, \qquad \dot{u}_i = \dot{x}_i, \qquad b_i = \frac{1}{k}E_i.$$

Explain then how the following gauge choice for the electromagnetic potentials,

$$\chi(\vec{r}) = \frac{1}{2}B\epsilon_{ij}\,x_i b_j \;+\; \bar{\chi}, \qquad q\Phi_0 = \frac{1}{2}k\vec{b}^{\,2},$$

where $\bar{\chi}$ is an arbitrary constant, allows one to finally bring the Lagrangian of the system in the following simple form,

$$L = \frac{1}{2}m\dot{u}_i^2 \;-\; \frac{1}{2}ku_i^2 \;-\; \frac{1}{2}B\epsilon_{ij}\,\dot{u}_i u_j.$$

2.3. The Hamiltonian formulation of the system then readily follows from the above Lagrangian. The momenta canonically conjugate to u_i are denoted π_i, with the Poisson brackets $\{u_i, \pi_j\} = \delta_{ij}$. Explain why the canonical Hamiltonian of the system is then given as,

$$H = \frac{1}{2m}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}u_j\right)^2 \;+\; \frac{1}{2}ku_i^2 = \frac{1}{2m}\pi_i^2 \;+\; \frac{1}{2}m\omega^2 u_i^2 \;-\; \frac{1}{2}\omega_c\,\epsilon_{ij}u_i\pi_j,$$

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[1] Further and relevant discussion of this system may be found in, Jan Govaerts, M. Norbert Hounkonnou and Habatwa V. Mweene, *J. Phys. A: Math. Theor.* **42** (2009) 485209 (19pp), [e-print: `arXiv:0909.2659 [hep-th]`].

where

$$\omega=\sqrt{\frac{k}{m}+\frac{1}{4}\omega_c^2}>0,\qquad \omega_c=\frac{B}{m}>0.$$

In its last form, this expression for the Hamiltonian is indeed very suggestive, in view of the results established in Problem 1. of this Assignment.

2.4. Canonical quantisation of the system then proceeds straightforwardly using the results of Problem 1. By introducing the relevant Fock operators $(a_i, a_i^\dagger)$ related this time to the operators $(\hat{u}_i, \hat{\pi}_i)$, and next the corresponding helicity ones $(a_\pm, a_\pm^\dagger)$, show that the quantum Hamiltonian operator of the system reduces to

$$\hat{H}=\hbar\omega_-\,a_+^\dagger a_+\;+\;\hbar\omega_+\,a_-^\dagger a_-\;+\;\hbar\omega,\qquad \omega_\pm=\omega\pm\frac{1}{2}\omega_c.$$

2.5. Identify then explicitly the energy eigenspectrum of the system, namely the energy eigenvalues and eigenstates. Draw in a diagram of which the horizontal axis is the variable $(n_+ - n_-)$ and the vertical one the energy value, the spectrum of energy states, where n_+ (resp., n_-) is the eigenvalue of the number operator $a_+^\dagger a_+$ (resp., $a_-^\dagger a_-$).

2.6. In the Heisenberg picture, solve for the time dependence of the operators $a_\pm$ and $a_\pm^\dagger$ and identify the time dependence of the position observables $\hat{x}_i(t)$ $(i=1,1)$. Infer then what the solutions to the dynamics are for the classical coordinates $x_i(t)$ $(i=1,2)$.

2.7. Consider now the system in the limit where, first, the electric field is taken away $(\vec{E}=\vec{0})$, and next, the harmonic potential $(k=0)$. How does the energy spectrum then look like? Describe the energy degeneracies that you observe (these are known as "the Landau levels" of the Landau problem). Show that in this limit, in the Heisenberg picture, the position operators are now of the form $\hat{x}_i(t)=\hat{x}_i^c\,+\,\hat{x}_i^{\rm circ}(t)$, with the following magnetic centre coordinates,

$$\hat{x}_i^c=\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+\,+\,a_+^\dagger\right),\qquad \hat{x}_2^c=i\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+\,-\,a_+^\dagger\right),$$

and the following coordinates of the circular trajectory (about the static magnetic centre),

$$\hat{x}_1^{\rm circ}(t)=\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\,e^{-i\omega_c t}\,+\,a_-^\dagger\,e^{i\omega_c t}\right),\qquad \hat{x}_2^{\rm circ}(t)=-i\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\,e^{-i\omega_c t}\,-\,a_-^\dagger\,e^{i\omega_c t}\right).$$

What are the commutation relations of these different coordinates? In particular note how the magnetic center coordinates define a noncommutative geometry in the euclidean plane.

2.8. There must exist an underlying explanation for the infinite degeneracies of the Landau levels in the absence of both the electric field and the harmonic potential. Since this degeneracy is clearly related to magnetic center sector of the system which may be positioned anywhere inside the plane, presumably it is translation invariance of the Landau problem which is the symmetry which accounts for this degeneracy. Indeed, the Lagrange function of the system, which now reads

$$L=\frac{1}{2}m\dot{x}_i^2\,-\,\frac{1}{2}B\epsilon_{ij}\,\dot{x}_i x_j,$$

is then invariant up to a surface term under constant translations in the plane, $\vec{r}\,'=\vec{r}+\vec{a}$, $\vec{a}$ being an arbitrary constant vector. Explain why the corresponding conserved Noether charge is given as $P_i=\pi_i\,-\,\frac{1}{2}B\,\epsilon_{ij}x_j$, π_i being the momenta conjugate to the coordinates x_i.

As operators, show that these generators of translations in the plane indeed commute with $\hat{H}$, hence define a symmetry of the quantum system, of which the algebra is (this algebra is commutative when $B=0$),

$$\left[\hat{P}_i,\hat{P}_j\right]=-i\hbar\,B\,\epsilon_{ij}\,\mathbb{I}.$$

Establish that as a matter of fact one has $\hat{P}_i=-B\epsilon_{ij}\,\hat{x}_j^c$, showing that up to normalisation, the magnetic centre coordinates are indeed the generators of translations in the plane, are conjugate to one another, which is why they define a noncommutative geometry in the plane, and map all quantum states belonging to a same Landau level into one another hence explaining the existence of the infinite degeneracies of these energy levels.