Assignment Sheet: Markov Chains

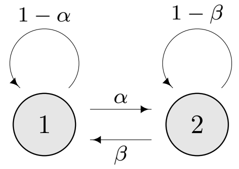
Please hand in your answers to at least four of the questions before Tuesday, March 2nd, 11.59pm. For the programming questions you can use whatever programming language you like (R, Pyhton, etc.). Please hand in your answers to programming questions (7 and 8) electronically by emailing your code to Madeleine (`madeleine@aims.ac.za`).

1. Show that a stochastic process in discrete time and having a discrete state space is a Markov chain if and only if for all $k \in \mathbb{N}$ and all $t_1 < \ldots < t_k \leq t$

$$\mathbb{P}(X_{t+1} = x_{t+1} | X_{t_k} = x_{t_k}, \ldots, X_{t_1} = x_{t_1}) = \mathbb{P}(X_{t+1} = x_{t+1} | X_{t_k} = x_{t_k}).$$

2. Let X be a Markov chain with state space $S = \{1, 2\}$ and transition kernel

$$\mathbf{K} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}, \qquad \text{where } \alpha, \beta \in (0,1)$$

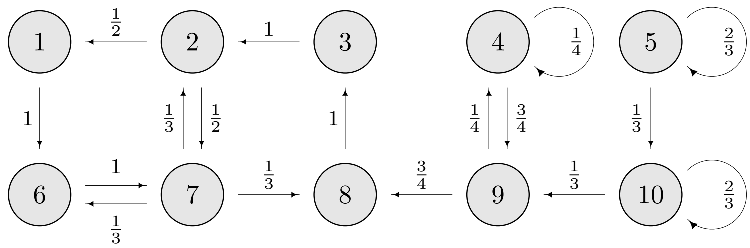


   (a) Show that $K^{(m)} = \begin{pmatrix} \frac{\beta + \alpha(1-\alpha-\beta)^m}{\alpha+\beta} & \frac{\alpha - \alpha(1-\alpha-\beta)^m}{\alpha+\beta} \\ \frac{\beta - \beta(1-\alpha-\beta)^m}{\alpha+\beta} & \frac{\alpha + \beta(1-\alpha-\beta)^m}{\alpha+\beta} \end{pmatrix}$

   (b) Find the invariant distribution of X.

   (c) Show that X is time-reversible if X_0 is drawn from the invariant distribution.

3. Consider a Markov chain with the following Markov graph:



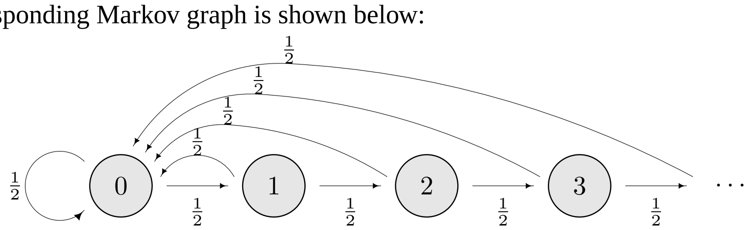
   (a) Identify the communicating classes.

   (b) For each class determine whether it is recurrent or transient, and whether it is periodic. If the latter is the case, determine the period.

4. Consider the Markov chain with state space $S = \mathbb{N}_0$ and transition probabilities

$$\mathbb{P}(X_{t+1} = 0 | X_t = i) = \frac{1}{2} \qquad\qquad \mathbb{P}(X_{t+1} = i+1 | X_t = i) = \frac{1}{2}$$

   for all $i \in \mathbb{N}_0$. The corresponding Markov graph is shown below:



   (a) Show that the chain is irreducible.

   (b) Show that all states are recurrent and aperiodic.

   (c) Find the invariant distribution of the chain.

5. Suppose you are gambling together with a friend. Your initial capital is £$A \in \mathbb{N}_0$, and your friend's is £$B \in \mathbb{N}_0$. Each time you toss a fair coin. If you win, you get £1 from your friend. If you lose, you give £1 to your friend. You and your friend only stop gampling if either you or your friend have run out of money. Denote by X_t your funds after the t-th game, where $X_0 = A$.

   (a) What is the state space of the Markov chain X? Draw the Markov graph of X.

   (b) Explain, using the concepts of recurrence and transience, that the game will not last forever, i.e. after some finite time either you or your friend will be bankrupt.

   (c) In this part of the question we will compute the probability that you will eventually end up loosing all your money, i.e. the probability of the event
   $$H_t = \{X_\tau = 0 \text{ for some } \tau > t\}.$$
   Denote by $h_i = \mathbb{P}(H_t|X_t = i)$ the probability of eventually ending up bankrupt if your current capital is £i.

   Show that
   $$h_0 = 1, \qquad h_i = \frac{1}{2}h_{i-1} + \frac{1}{2}h_{i+1} \text{ for } i \in \{1, \ldots, A+B-1\}, \qquad h_{A+B} = 0,$$
   and derive from this that
   $$h_i = \frac{A+B-i}{A+B}.$$
   Thus, at the begin of the game, what is the probability that you will loose all your money?

6. Let X be a Markov chain with discrete state space S. Then let
$$S_t = \min\{\tau \geq 1 : \; X_{t+\tau} \neq X_t\}$$
be the time until chain leaves its current state X_t for the first time.

   (a) Show that the probability mass function of S_t given that $X_t = x_t$ is
   $$p(s|X_t = x_t) = \begin{cases} (1 - k_{x_t x_t})(k_{x_t x_t})^{s-1} & \text{for } s \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Show that the distribution of S_t is memoryless, i.e.
   $$\mathbb{P}(S_t > s_0 + s|S_t > s_0) = \mathbb{P}(S_t > s).$$

7. In a study on social mobility in the United Kingdom in the 1950s (a time of low unemployment), the occupational status of 3498 fathers and their (oldest) sons was recorded. The eight categories used were as follows.

1	Professional and high administrative
2	Managerial and executive
3	Inspectory, supervisory and other non-manual (high grade)
4	Inspectory, supervisory and other non-manual (low grade)
5	Routine grades of non-manual
6	Skilled manual
7	Semi-skilled manual
8	Unskilled manual

The cross-classification table below contains the number of pairs of fathers and sons for each combination of occupations:

Occupational status of the father \ Occupational status of the son	1	2	3	4	5	6	7	8
1	50	19	26	8	7	11	6	2
2	16	40	34	18	11	20	8	3
3	12	35	65	66	35	88	23	21
4	11	20	58	110	40	183	64	32
5	2	8	12	23	25	46	28	12
6	12	28	102	162	90	554	230	177
7	0	6	19	40	21	158	143	71
8	0	3	14	32	15	126	91	106

Carry out tasks (a) to (c) on a computer (using e.g. R oder Python).

   (a) Compute the estimated (marginal) distribution of occupations both for the fathers and the sons.

(b) If one records the occupational status of the father, his son, his grandson, etc., one can model their occupational status as a Markov chain. Based on the table given above, estimate the transition kernel using

$$\hat{k}_{ij} = \frac{\text{number of sons in category } j \text{ having a father in categroy } i}{\text{number of fathers in category } i}$$

(c) Compute the invariant distribution of the Markov chain.

(d) Compare the invariant distribution found in part (c) to the distribution of fathers and sons.

8. A simple stochastic model for reproduction in population genetics is the Wright-Fisher model.

Suppose that at a given location on the gene, only two different alleles are possible: we shall call them A_1 and A_2. As humans are diploid (we have two copies of each chromosome), the number of relevant chromosomes is a population of size N is thus $2N$.

The Wright-Fisher model makes the following assumptions:

- The population size N is finite.
- Mating is random with the respect to the gene being studied, i.e. neither A_1, nor A_2 make you fitter or more attractive.
- Generations are not overlapping. (Otherwise the resulting process would not be a Markov chain any more.)

If we denote by X_t the number of alleles A_1 in generation t, then

$$\mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_t) = \binom{2N}{x_{t+1}} \cdot \left(\frac{x_t}{2N}\right)^{x_{t+1}} \left(\frac{2N - x_t}{2N}\right)^{2N - x_{t+1}}$$

i.e. $X_{t+1} | X_t = x_t \sim \mathsf{Bi}\left(2N, \frac{x_t}{2N}\right)$

Essentially, the Wright-Fisher model assumes that each child randomly picks two chromosomes from the previous generation (thus children choose their parents).

Write a function (e.g. in R or Python) that takes the initial counts for both alleles (i.e. X_0 and $2N - X_0$) as arguments, and then simulates 100 generations from the Wright-Fisher model.

For one realisation, create a plot of X_t (and $2N - X_t$) over time.

Hint: The R function `rbinom` *and the Python function* `numpy.random.binomial` *sample from the binomial distribution.*