Operators and their Applications to Nonlinear Elliptical Boundary Value Problems

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Abstract

The modern approach to boundary-value problems governed by partial differential equations starts with the introduction of a weak formulation of the problem whose solution is obtained through an abstract theory of operators.

For some non-linear boundary value problems, the weak formulation induces a non-linear mapping between Banach spaces. The results of existence and uniqueness of the weak or generalized solution of the boundary value problem are then obtained from the properties of pseudomonotone and coercivity of the non-linear mapping.

The purpose of this project is to explore the theory of monotone operators between Banach spaces and apply them to the study of non-linear boundary value problems.

The main tools used for the second part of the project are the L^p and Sobolev spaces. The project requires having a good background in functional analysis and function spaces.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Gezahegn Zewdie Abebe, 20 May 2010

Contents

Abstract **i**

1 Introduction **1**

2 Preliminaries **2**

2.1 Banach Spaces 2

3 Operators **4**

3.1 Monotone Operators 4

3.2 Hemicontinuity and Demicontinuity 10

3.3 Pseudomonotone Operators 11

3.4 Convex sets 12

3.5 Convex functions 12

4 Application of Operators **15**

4.1 L^p Spaces 15

4.2 Sobolev Spaces 16

4.3 Elliptic Boundary Value Problem 19

5 Conclusion **26**

References **28**

1. Introduction

In this essay, we will show the existence of solution for non-linear boundary value problems governed by non-linear second order elliptical equations. Functional spaces and their properties will be discussed briefly. These functional spaces are *Banach spaces*, L^p *spaces* and *Sobolev spaces*. The L^p and the *Sobolev spaces* are the most important tools for all of the work in this essay.

In the second chapter, we will have a short description of Banach space and its dual.

Operators and their properties will be discussed in the third chapter. These are monotone and pseudomonotone operators. In Chapter 4, we will talk about *Coercivity of monotone operators* and *pseudomonotonincity operator*, which are the prime property for the expected work in chapter four.

Pseudomonotonincity and coercivity operator from a reflexive and separable normed space to its dual has an essential application in the project. Non-linear boundary value problems having a weak formulation and which can be written weak formulation which can be written can be written in an operational form.

Finally, by showing pseudomonotonicity and coercivity, we obtain the existence of at least one solution of the boundary value problem.

2. Preliminaries

2.1 Banach Spaces

Definition 2.1.1. *Let X be a vector space over a scalar field $\mathbb{R}$. A norm on X is a function $||\cdot|| : X \longrightarrow \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{R}$[Win],*

i. $||x|| \geq 0$ *and* $||x|| = 0$ *if and only if* $x = 0$*;*

ii. $||\alpha x|| = |\alpha| ||x||$*;*

iii. $||x + y|| \leq ||x|| + ||y||$ *(triangle inequality).*

In addition, the pair $(X, ||\cdot||)$ is called a normed (vector) space.

A mapping L of a vector space X in to a vector space Y is called linear operator, linear mapping or linear transformation over some field $\mathbb{K}$ if,

$$L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2) \tag{2.1}$$

where $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{K}$.

Remark 2.1.2. *A linear operator $L : X \longrightarrow Y$ is said to be bounded if there exists M such that $||L(x)|| \leq M||x||$ for all $x \in X$ provided that X and Y are normed spaces.*

Definition 2.1.3. *An operator $A : X \longrightarrow Y$ is said to be non linear if it is not linear.*

Definition 2.1.4. *[Roy63] A normed linear space is called complete if every Cauchy sequence in the space converges.*

A complete normed linear space is called a Banach Space.

Stefan Banach (1892-1945), Polish Mathematician,

Example 2.1.5. *The Euclidean spaces $\mathbb{R}^n, n > 0$, where the Euclidean norm of $x = (x_1, \ldots, x_n)$, is given by $||x|| = (\sum_{i=1,\ldots,n} |xi|^2)^{1/2}$, are Banach spaces.*

The set of bounded linear operators from X to Y has a structure of vector space and denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}(X)$ if $Y = X$.

Definition 2.1.6. *Let V be a normed vector space over a field $\mathbb{K}$, the space $\mathcal{L}(X, \mathbb{K})$ is called a dual space of V and denoted by V^*. The elements of V^* are called bounded linear functionals on V.*

$V^{**} := (V^*)^*$ is called a bidual space of V.

Definition 2.1.7. *A normed space V is said to be reflexive if $V = V^{**}$*

Proposition 2.1.8. *Let X and Y be two normed spaces. If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.*

Definition 2.1.9. *Let V be a normed space and V^* its dual. A sequence v_n in V is said to converge weakly in V to v if*

$$\lim_{n\to\infty} \langle \ell, v_n \rangle = \langle \ell, v \rangle \text{ for all } \ell \in V^* \tag{2.2}$$

Weak convergence is usually denoted by $x_n \rightharpoonup x$.

Strong (norm) convergence implies weak convergence, but the converse does not hold.

3. Operators

Monotone and pseudomonotone operators are the important tools for solving non-linear elliptical boundary value problems. Properties of monotone operators and relation between them will be discussed in this chapter. Coercivity of a monotone operators is the crucial property over all work.

3.1 Monotone Operators

Definition 3.1.1. *[Zei85a] Let X and Y be real Banach spaces, and $A : X \longrightarrow X^*$ an operator*

i. A is called monotone if, and only if,

$$\langle A(u) - A(v), u - v\rangle \geq 0, \text{ for all } u, v \in X. \tag{3.1}$$

ii. A is called strictly monotone if, and only if,

$$\langle A(u) - A(v), u - v\rangle > 0, \text{ for all } u, v \in X \text{ with } u \neq v. \tag{3.2}$$

iii. A is called strongly monotone if, and only if, there exist $c > 0$ such that

$$\langle A(u) - A(v), u - v\rangle \geq c||u - v||^2 \text{ for all } u, v \in X. \tag{3.3}$$

iv. A is called uniformly monotone if, and only if, there exist a such that

$$\langle A(u) - A(v), u - v\rangle \geq a(||u - v||)||u - v|| \text{ for all } u, v \in X. \tag{3.4}$$

where the continuous function $a : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$

v. A is called coercive if

$$\lim_{||u||\to\infty} \frac{\langle A(u), u\rangle}{||u||} = +\infty. \tag{3.5}$$

vi. A is called weakly coercive if

$$\lim_{||u||\to\infty} ||A(u)|| = +\infty. \tag{3.6}$$

vii. An operator $A : X \longrightarrow Y$ is called stable if, and only if,

$$||A(u) - A(v)|| \geq a(||u - v||) \text{ for all } u, v \in X, \tag{3.7}$$

where the continuous function $a : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$.

Theorem 3.1.2. *Let $A : X \longrightarrow X^*$ be an operator*

i. If A is strongly monotone then A is uniformly monotone.

ii. If A is uniformly monotone then A is strictly monotone.

iii. *If A is strongly monotone then A is strictly monotone.*

iv. *If A is strictly monotone then A is monotone .*

v. *If A is uniformly monotone then A is coercive and stable.*

Proof. i. A is strongly monotone if, and only if,

$$\begin{aligned}\langle A(u) - A(v), u - v\rangle &\geq c||u - v||^2 \textit{ for all } \quad u, v \in X, c > 0 \\ &= c||u - v||||u - v||\end{aligned} \tag{3.8}$$

Now choose $a(t) = c|t|$ and we have

$$c||u - v||||u - v|| = a(||u - v||)||u - v||.$$

Therefore, A is uniformly monotone.

ii. A be uniformly monotone if, and only if,

$$\begin{aligned}&\langle A(u) - A(v), u - v\rangle \\ &= a(||u - v||)||u - v|| \textit{ for all } \quad u, v \in X,\end{aligned} \tag{3.9}$$

and a be a continuous function, implies

$$a(||u - v||)||u - v|| > 0 \quad \textit{whenever} \quad u \neq v, \tag{3.10}$$

Hence, the operator A is strictly monotone.

iii. A is strongly monotone if, and only if,

$$\begin{aligned}&\langle A(u) - A(v), u - v\rangle \\ &= c||u - v||^2 \quad \textit{for all} \quad u, v \in X \quad \textit{and for} \quad c > 0 \\ &> 0 \quad \textit{whenever} \quad u \neq v.\end{aligned} \tag{3.11}$$

Hence, A is strictly monotone.

iv. If A is strictly monotone then obviously, A is monotone.

v. Let an operator A is uniformly monotone then, there exist a function a such that

$$\begin{aligned}\langle A(u), u\rangle &= \langle A(u) - A(0), u - 0\rangle \\ &\geq a(||u - 0||)||u - 0|| \\ &= a(||u||)||u||.\end{aligned} \tag{3.12}$$

This implies that

$$\frac{\langle A(u), u\rangle}{||u||} \geq a(||u||). \tag{3.13}$$

By taking the limit at ∞ of both sides we have

$$\lim_{||u|| \to \infty} \frac{\langle A(u), u\rangle}{||u||} \geq \lim_{||u|| \to \infty} a(||u||) = \infty. \tag{3.14}$$

Hence, A is coercive.

To show A is stable

$$||A(u) - A(v)||||u - v|| \geq \langle A(u) - A(v), u - v\rangle \tag{3.15}$$

since A is uniformly monotone,

$$\langle A(u) - A(v), u - v\rangle \geq a(||u - v||)||u - v||, \tag{3.16}$$

implies

$$||A(u) - A(v)|| \geq a(||u - v||). \tag{3.17}$$

Hence, A is stable.

□

Theorem 3.1.3 (Linear monotone operators)**.** *[Zei85b] Let $A : X \longrightarrow X^*$ be a linear operator on the real Banach space then,*

i. A is monotone if, and only if, A is positive, i.e, $\langle A(u), u\rangle \geq 0$ for all $u \in X$.

ii. A is strictly monotone if, and only if, A is strictly positive, i.e, $\langle A(u), u\rangle > 0$ for all $u \in X$.

iii. A is strongly monotone if, and only if, A is strongly positive, i.e, $\langle A(u), u\rangle \geq c||u||^2$for all $u \in X$ and $c > 0$

Proof. i. A is monotone if, and only if,

$$\langle A(u) - A(v), u - v\rangle \geq 0 \text{ for } u, v \in X \tag{3.18}$$

since $A(u) - A(v) = A(u - v)$,

$$\langle A(u) - A(v), u - v\rangle = \langle A(u - v), u - v\rangle \geq 0, \tag{3.19}$$

Hence,

$$\langle A(w), w\rangle \geq 0, \text{ for } u - v = w \in X. \tag{3.20}$$

Therefore, A is positive.

ii. A is strictly monotone if, and only if,

$$\langle A(u) - A(v), u - v\rangle > 0 \text{ for } u, v \in X \tag{3.21}$$

since $A(u) - A(v) = A(u - v)$,

$$\langle A(u) - A(v), u - v\rangle = \langle A(u - v), u - v\rangle > 0, \tag{3.22}$$

Hence,

$$\langle A(w), w\rangle > 0, \text{ for } u - v = w \in X. \tag{3.23}$$

Therefore, A strictly is positive.

iii. A is strongly monotone if, and only if,

$$\langle A(u) - A(v), u - v\rangle \geq c||u - v|| \text{ for } u, v \in X \tag{3.24}$$

since $A(u) - A(v) = A(u - v)$,

$$\langle A(u - v), u - v\rangle \geq c||u - v||. \tag{3.25}$$

Hence,

$$\langle A(w), w\rangle \geq c||w||, \text{ where } w = u - v. \tag{3.26}$$

Therefore, A strongly is positive.

□

Theorem 3.1.4 (Monotone Real Function). *[Zei85b] Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and regard f as an operator from $X \longrightarrow X^*$ with $X = \mathbb{R}$ such that $\langle f(u) - f(v), u - v\rangle = (f(u) - f(v))(u - v)$ for $u, v \in \mathbb{R}$ then,*

i. $f : X \longrightarrow X^*$ is (strictly) monotone if, and only if, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is (strictly) monotone increasing.

ii. $f : X \longrightarrow X^*$ is strongly monotone if, and only if,

$$\inf_{u \neq v} \frac{f(u) - f(v)}{u - v} > 0.$$

iii. $f : X \longrightarrow X^*$ is coercive if, and only if,

$$\lim_{u \to \pm\infty} f(u) = \pm\infty.$$

iv. If $F : \mathbb{R} \longrightarrow \mathbb{R}$ is C^2 satisfying $F''(u) \geq c$ for all $u \in \mathbb{R}$ and $c > 0$ then,

$$(F'(u) - F'(v))(u - v) \geq c(u - v)^2$$

for all $u, v \in \mathbb{R}$.

That means F' is strongly monotone.

v. If $F : \mathbb{R} \longrightarrow \mathbb{R}$ is C^1 satisfying

$$F'(u) - F'(v) \geq c(u - v),$$

for all $u, v \in \mathbb{R}$ with $u \geq v$ and $c > 0$, then, $F' : \mathbb{R} \longrightarrow \mathbb{R}$ is strongly monotone .

Proof. i. f is monotone if, and only if,

$$\langle f(u) - f(v), u - v\rangle \geq 0, u, v \in \mathbb{R} \tag{3.27}$$

Since $\langle f(u) - f(v), u - v\rangle = (f(u) - f(v))(u - v)$, we have

$$(f(u) - f(v))(u - v) \geq 0 \tag{3.28}$$

then

$$(f(u) - f(v)) \geq 0 \textit{ and } (u - v) \geq 0 \tag{3.29}$$

or

$$(f(u) - f(v)) \leq 0 \textit{ and } (u - v) \leq 0. \tag{3.30}$$

Hence, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone increasing.

Similarly $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly monotone increasing

ii. $f : X \longrightarrow X^*$ is strongly monotone if, and only if,

$$\langle f(u) - f(v), u - v\rangle \geq c||u - v||^2, \textit{ for } c > 0. \tag{3.31}$$

Since

$$\langle f(u) - f(v), u - v\rangle = (f(u) - f(v))(u - v) \tag{3.32}$$

$$(f(u) - f(v))(u - v) \geq c||u - v||^2, \tag{3.33}$$

equivalently

$$(f(u) - f(v))(u - v) \geq c(u - v)(u - v) \tag{3.34}$$

then

$$\frac{f(u) - f(v)}{u - v} \geq c, u \neq v \tag{3.35}$$

we have

$$\inf_{u \neq v} \frac{f(u) - f(v)}{u - v} > 0, \textit{ since } c > 0, u \neq v. \tag{3.36}$$

iii. $f : X \longrightarrow X^*$ is coercive if, and only if,

$$\lim_{||u|| \to \infty} \frac{\langle f(u), u\rangle}{||u||} = +\infty. \tag{3.37}$$

Since

$$\langle f(u), u\rangle = f(u)(u) \tag{3.38}$$

$$\lim_{||u|| \to \infty} \frac{f(u)(u)}{||u||} = +\infty \tag{3.39}$$

and we have

$$\lim_{u \to \pm\infty} f(u) = \pm\infty \textit{ since } u \in \mathbb{R}. \tag{3.40}$$

iv. Since $F''(u) \geq c$ where $c > 0$, we have:

$$F''(u) = \frac{F'(u) - F'(v)}{u - v} \geq c \tag{3.41}$$

implies

$$\frac{F'(u) - F'(v)}{(u - v)^2}(u - v) \geq c(F'(u) - F'(v))(u - v) \geq c(u - v)^2 \quad \textit{for all} \quad u \neq v. \tag{3.42}$$

Hence, F' is strictly monotone.

v. We are given that $F'(u) - F'(v) \geq c(u-v)$ which implies

$$(F'(u) - F'(v))(u-v) \geq c(u-v)^2 \quad \textit{for all} \quad u \neq v \tag{3.43}$$

Hence, F' is strictly monotone.

□

[Zei85a] The following theorem shows that the monotone operator $A : X \longrightarrow X^*$ is characterised by monotone function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Theorem 3.1.5. *Let $A : X \longrightarrow X^*$ be an operator on the real Banach space X. We set*

$$f(t) = \langle A(u+tv), v\rangle \quad \textit{for all } t \in \mathbb{R}$$

Then the following are equivalent:

i. The operator A is monotone

ii. The function $f : [0,1] \longrightarrow \mathbb{R}$ is monotone increasing.

Proof. Suppose $A : X \longrightarrow X^*$ be a monotone operator and let $0 \leq s < t$,

$$f(t) = \langle A(u+tv), v\rangle \quad \textit{and } f(s) = \langle A(u+sv), v\rangle \tag{3.44}$$

implies

$$\begin{aligned} f(t) - f(s) &= \langle A(u+tv), v\rangle - \langle A(u+sv), v\rangle \\ &= \langle A(u+tv) - A(u+sv), tv - sv\rangle (t-s)^{-1} \\ &= \langle A(u+tv) - A(u+sv), (u+tv) - (u+sv)\rangle (t-s)^{-1}. \end{aligned} \tag{3.45}$$

Hence,

$$\langle A(u+tv) - A(u+sv), (u+tv) - (u+sv)\rangle (t-s)^{-1} \geq 0 \quad . \tag{3.46}$$

$f : [0,1] \longrightarrow \mathbb{R}$ is monotone increasing.

This implies that

$$f(t) - f(s) \geq 0. \tag{3.47}$$

Hence, f is monotone increasing

Conversely, suppose $f : [0,1] \longrightarrow \mathbb{R}$ is monotone increasing. Since,

$$f(t) = \langle A(u+tv), v\rangle \quad \textit{for all } t \in \mathbb{R} \tag{3.48}$$

$$f(0) = \langle A(u), v\rangle \tag{3.49}$$

and

$$f(1) = \langle A(u+v), v\rangle. \tag{3.50}$$

Now,

$$\begin{aligned} &\langle A(u+v) - A(u), u+v-u\rangle \\ &= \langle A(u+v), v\rangle - \langle A(u), v\rangle \\ &= f(1) - f(0) \geq 0, \end{aligned} \tag{3.51}$$

implies

$$\langle A(u+v) - A(u), u+v-u \rangle \geq 0. \tag{3.52}$$

Hence, A is monotone operator. □

Theorem 3.1.6 (Banach-Steinhaus theorem)**.** *Let X be a Banach space and Y be a normed vector space. Suppose that F is a collection of continuous linear operators from X to Y. The uniform boundedness principle states that if for all x in X we have*

$$\sup_{T \in F} ||T(x)|| < \infty \tag{3.53}$$

then,

$$\sup_{T \in F} ||T|| < \infty. \tag{3.54}$$

3.2 Hemicontinuity and Demicontinuity

Definition 3.2.1. *[Zei85a] Let $A : X \longrightarrow X^*$ be an operator on real Banach space*

i. A is said to be $demicontinuous$ if, and only if, $u_n \longrightarrow u$ as $n \longrightarrow \infty$ implies $A(u_n) \rightharpoonup A(u)$ as $n \longrightarrow \infty$.

ii. A is said to be $hemicontinuous$ if, and only if, the real function

$$t \longmapsto \langle A(u+tv), w \rangle$$

is continuous on $[0,1]$ for all $u, v, w \in X$.

iii. A is said to be $strongly\ continuous$ if, and only if, $u_n \rightharpoonup u$ as $n \longrightarrow \infty$ implies $A(u_n) \longrightarrow A(u)$ as $n \longrightarrow \infty$.

iv. A is said to be $bounded$ if, and only if, A maps bounded sets into bounded sets.

Theorem 3.2.2. *Let $A : X \longrightarrow X^*$ be an operator on a real Banach space X. Then,*

i. If A is monotone , A is locally bounded.

ii. If A is linear and monotone , A is continuous.

iii. If A is monotone and hemicontinuous on a real reflexive Banach space X, then, A is demicontinuous.

Proof. i. Let A be monotone and assume A is not locally bounded then there exists a point $u \in X$ and the sequence (u_n) with

$$u_n \longrightarrow u$$

and

$$||A(u_n)|| \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

Without loss of generality, let $u = 0$. We set,

$$a_n = (1 + ||A(u_n)|| ||u_n||)^{-1}.$$

From the monotonicity of the operator A it follows that

$$\begin{aligned} \pm a_n\langle A(u_n), v\rangle &\le a_n(\langle A(u_n), u_n\rangle - \langle A(\pm v), u_n \mp v\rangle) \\ & a_n(||A(u_n)||||u_n|| + ||A(\pm v)||||u_n \mp v||). \end{aligned} \tag{3.55}$$

Therefore,

$$\sup|\langle a_n A(u_n), v\rangle| < \infty \quad \textit{for all } v \in X \tag{3.56}$$

since by Banach-Steinhaus theorem.

$$\sup||a_n A(u_n)|| \le N \quad \textit{for } N \in \mathbb{N}\ , \tag{3.57}$$

We set

$$b_n = ||A(u_n)||. \tag{3.58}$$

Then,

$$b_n \le a_n^{-1}N = (1 + b_n||(u_n)||)N, \textit{for all } n. \tag{3.59}$$

And since

$$||u_n|| \longrightarrow 0 \quad \textit{as } n \longrightarrow \infty, \tag{3.60}$$

the sequence b_n is bounded and contradict with A not locally bounded.

Hence, $||A(u_n)|| \longrightarrow \infty$ as $n \longrightarrow \infty$

ii. From a), the fact is that a linear locally bounded operator A is bounded on a neighbourhood of zero and thus also on the closed unit ball,

$$||A(u)|| \le c||u|| \textit{ for some } c \in \mathbb{N},\ u \in X. \tag{3.61}$$

Hence, A is continuous

iii. Let $u_n \longrightarrow u$ as $n \longrightarrow \infty$. Since u_n is bounded, the sequence $(A(u_n))$ is also bounded in (a). Let $A(u_{n'}) \longrightarrow b$ *as* $n \longrightarrow \infty$ *for a sequence* $(u_{n'})$ *of* (u_n). Then,

$$\begin{aligned} &\langle A(u_{n'}), u_{n'}\rangle \longrightarrow \langle b, u\rangle \quad \textit{as } n \longrightarrow \infty \\ &A(u) = b \textit{ by monotonicity.} \end{aligned} \tag{3.62}$$

Hence, by principle of convergence, $A(u_n) \longrightarrow A(u)$ *as* $n \longrightarrow \infty$

□

3.3 Pseudomonotone Operators

Definition 3.3.1. *[Zei85a] A mapping* $A : X \longrightarrow X^*$ *is called* $pseudomonotone$ *if, and only if,* A *is bounded and*

$$u_n \rightharpoonup n \quad \textit{as} \quad n \longrightarrow \infty \quad \textit{and} \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0. \tag{3.63}$$

implies

$$\langle A(u), u - v\rangle \le \liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \quad \textit{for all } v \in X. \tag{3.64}$$

Theorem 3.3.2. *Let $A, B : X \longrightarrow X^*$ be operators on the real reflexive Banach space. Then,*

i. *If A is monotone and hemicontinuous, then A is pseudomonotone.*

ii. *If A is strongly continuous, then A is pseudomonotone.*

iii. *If A and B are pseudomonotone , then $A + B$ is also pseudomonotone.*

iv. *If A is monotone and hemicontinuous and B is strongly continuous, then $A+B$ is pseudomonotone.*

3.4 Convex sets

Definition 3.4.1. *A convex set is a set of elements from a vector space such that all the points on the straight line between any two points of the set are also contained in the set. If a and b are points in a vector space the points on the straight line between a and b are given by*

$$x = \lambda a + (1 - \lambda)b \qquad \text{for all} \qquad \lambda \in [0, 1]. \tag{3.65}$$

Convex sets in $\mathbb{R}$ are intervals.

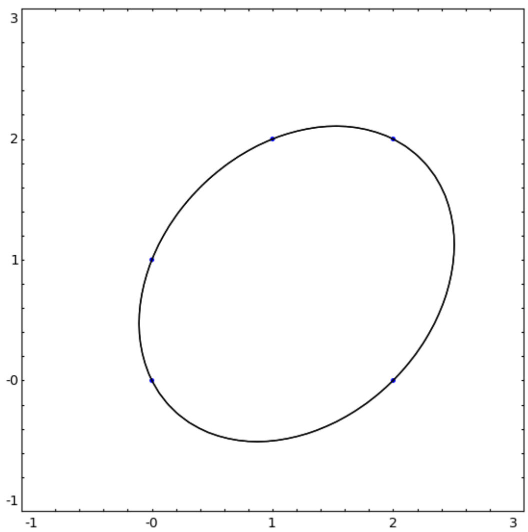

Figure 3.1: Convex set in $\mathbb{R}^2$

3.5 Convex functions

Definition 3.5.1. *A real-valued function f defined on an interval (or on any convex subset C of some vector space) is called convex if for any two points x and y in its domain C and any t in $[0, 1]$,*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y). \tag{3.66}$$

It is said to be strictly convex if

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y) \tag{3.67}$$

where C be any convex set or an interval.

Proposition 3.5.2. *Let $f : C \longrightarrow \mathbb{R}$ be a G-differentiable (Gateaux differentiable) function and C be a convex subset of B-space X, then f is strictly convex on C if, and only if, f' is strictly monotone on C.*

Theorem 3.5.3. *[Rou00] Any pseudomonotone and coercive operator A from a reflexive and separable normed space to its dual is surjective. That is for every $f \in V^*$ there exists at least one $u \in V$ such that $A(u) = f$.*

Proof. V is separable implies that there exist some vectors $\{w_1, w_2, ..., w_n, ...\}$ in V such that the subspace generated by the these vectors is dense. Let $V_n = \langle w_1, ..., w_n \rangle$ be the subspace generated by the first n vectors and $V_n \subseteq V_{n+1} \subseteq V$, $A(u) = f$ *iff* $\langle A(u), v \rangle = \langle f, v \rangle$ *for all* $v \in V$

$$\bigcup_n V_n \quad \textit{is dense in } V.$$

Consider $u_n \in V_n$ such that

$$\langle A(u_n), v \rangle = \langle f, v \rangle \quad \textit{for all} \quad v \in V_n. \tag{3.68}$$

From (3.68), we have

$$\langle A(u_n), w_k \rangle = \langle f, w_k \rangle \quad k = 1, 2, ... \tag{3.69}$$

$$\langle A(u) - f, u \rangle = \langle A(u), u \rangle - \langle f, v \rangle \geq \langle A(u), u \rangle - ||f||_{V^*} ||u||_V \tag{3.70}$$

When $||u||$ is large enough, $\frac{\langle A(u), u \rangle}{||u||} > ||f||_{V^*}$ implies

$$\sum_{k=1}^{n} \langle A(u_n) - f, w_k \rangle \, c_{n,k} > 0 \tag{3.71}$$

where $u_n = \sum_{k=1}^{n} c_{n,k} w_k$ has a solution u_n.

Let $u_n \subset V$ and be bounded. i.e. there exist ρ such that $u_n \subseteq V$, $||u_n|| \leq \rho$
and then $\langle A(u_n) - f, w_k \rangle = 0$ which implies $\{A(u_n)\}$ is bounded.

Since V is reflexive any bounded sequence in V has a weakly convergent subsequence $\{u_{n_k}\}$ of (u_n) such that $u_{n_k} \rightharpoonup u$. Now fix $m \in \mathbb{N}$ and let $n \geq m$ such that $V_m \subseteq V_n$

$$\langle A(u_n) - f, v \rangle = 0 \quad \textit{for all} \quad v \in V_n \tag{3.72}$$

for all $v_n \in V_m$

$$\langle A(u_n) - f, V_m - u_n \rangle = 0 \tag{3.73}$$

$$\begin{aligned} \langle A(u_n), u_n - u \rangle &= \langle A(u_n), u_n - v_n \rangle + \langle A(u_n), v_n - u \rangle \\ &\leq \langle f, u_n - v_n \rangle + ||A(u_n)||_{v^*} ||v_n - u|| \leq c||u_n - u|| \end{aligned} \tag{3.74}$$

implies

$$\lim_{n \to \infty} \sup \langle A(u_n), u_n - u \rangle \leq \lim_{n \to \infty} \langle f, u_n - v_n \rangle . \tag{3.75}$$

By definition of monotonicity

$$\langle A(u), u_n - u\rangle \leq \lim_{n\to\infty} \langle f, u_n - v_n\rangle . \tag{3.76}$$

In particular, let

$$v \in \bigcup_m V_m \quad \textit{and there exist} \quad n : v \in V_n$$

and for $n > m$ such that $v \in V_m \subset V_n$ then,

$$\langle A(u), u_n - v\rangle = \langle f, u_n - v\rangle \tag{3.77}$$

$$\langle A(u), u_n - u\rangle = \lim_{n\to\infty} \langle f, u_n - v_n\rangle = \langle f, u - v\rangle \tag{3.78}$$

$$\langle A(u), u - v\rangle \leq \langle f, u - v\rangle \qquad \textit{for all } v \in \bigcup_m V_m, \tag{3.79}$$

this implies

$$\langle A(u), \lambda w\rangle \leq \langle f, \lambda w\rangle \qquad \textit{for all } \lambda \in \mathbb{R} \tag{3.80}$$

the equality holds for if $\lambda = \pm 1$.

Hence, $A(u) = f$. □

4. Application of Operators

The basic modern approach to boundary-value problems in differential equation of the type

$$(P) \qquad \begin{cases} -\text{div } a(x, u(x), \triangledown u(x)) + c(x, u(x), \triangledown u(x)) = f(x) \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

the method is called energy-method, which relies on relative weak compactness of bounded sets in reflexive Banach spaces, and either pseudomonotonicity or weak continuity of suitable differential operators evolved from the boundary value problem [Rou00].

4.1 L^p Spaces

The L^p space can be defined a in general context of measurable space (X, Σ, μ) where Σ is σ-algebera on the set X and $\mu : \Sigma \longrightarrow [0, \infty)$ is a positive measure, we introduce the situation where Ω is bounded open subset of $\mathbb{R}^n$ and $p[1, \infty)$,

$$L^p = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } \int_\Omega |f|^p d\mu < \infty \right\} \tag{4.1}$$

Remark 4.1.1 (Young inequality). *Let $a, b > 0$ and $p, q \in (0, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ then,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 4.1.2 (Hölders inequality). *[Roy63] If p and q are extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p$ and $g \in L^q$, then*

$$\int |fg| \leq ||f||_p ||g||_q.$$

Theorem 4.1.3 (Generalised Hölder's inequality). *Let $p, q, r \in [1, \infty)$ and let $\alpha \geq 1$ such that*

$$\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

If $f \in L^p(\Omega), g \in L^q(\Omega)$ and $h \in L^h(\Omega)$ then, $fgh \in L^\alpha$ and $||fgh||_{L^\alpha} \leq ||h||_{L^r} ||f||_{L^p} ||g||_{L^q}$

Proof. Let $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ which implies $1 = \frac{1}{\frac{p}{\alpha}} + \frac{1}{\frac{q}{\alpha}} + \frac{1}{\frac{r}{\alpha}}$ for $\alpha \geq 1$; we have $\alpha(\frac{1}{p} + \frac{1}{q}) = \alpha(\frac{p+q}{pq}) = \frac{1}{\frac{pq}{\alpha(p+q)}} + \frac{1}{\frac{r}{\alpha}}$ and let $f \in L^p, g \in L^q$ and $h \in L^h$.

Now consider

$$\int_\Omega |fgh|^\alpha = \int_\Omega |gf|^\alpha |h|^\alpha \leq \left(\int_\Omega |h|^r \right)^{\frac{\alpha}{r}} \left(\int_\Omega |fg|^{\frac{pq}{p+q}} \right)^{\frac{\alpha(p+q)}{pq}}. \tag{4.2}$$

Let $s = \frac{p+q}{q}, s' = \frac{s}{s-1} = \frac{\frac{p+q}{q}}{\frac{p}{q}} = \frac{p+q}{p}$, then ,

$$\begin{aligned}\int_\Omega |gf|^{\frac{pq}{p+q}}dx &= \int_\Omega |f|^{\frac{pq}{p+q}}|g|^{\frac{pq}{p+q}}dx \\ &\le \left(\int_\Omega \left(|f|^{\frac{pq}{p+q}}\right)^s dx\right)^{\frac{1}{s}} \left(\int_\Omega \left(|g|^{\frac{pq}{p+q}}\right)^{s'} dx\right)^{\frac{1}{s'}} \\ &\le \left(\int_\Omega |f|^p dx\right)^{\frac{q}{p+q}} \left(\int_\Omega |g|^q dx\right)^{\frac{p}{p+q}} = ||f||_{L^p}^{\frac{pq}{p+q}} ||g||_{L^q}^{\frac{pq}{p+q}}.\end{aligned} \tag{4.3}$$

Hence,

$$\int_\Omega |fgh|^\alpha \le ||h||_{L^r}^\alpha ||f||_{L^p}^\alpha ||g||_{L^q}^\alpha \tag{4.4}$$

$$||fgh||_{L^\alpha} \le ||h||_{L^r} ||f||_{L^p} ||g||_{L^q} \tag{4.5}$$

and which implies $fgh \in L^\alpha$. □

Proposition 4.1.4. *[Ebo10]*

If the sequence $\{f_n\}_n$ converges to f in $L^p(\Omega)$, then

i. there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ which converges to f a.e. in Ω.

ii. there exists a function $g \in L_p(X, \lambda)$ such that $|f_{n_k}(x)| \le g(x)$ a.e. in X.

It is known through the Hölder's and Minkowski's inequalities the mapping

$$|| \cdot ||_p : L^p(\Omega) \longrightarrow [0, \infty] \tag{4.6}$$

defined by

$$||f||_f := \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} \tag{4.7}$$

is a norm and $(L^p(\Omega), ||\cdot||_p)$ is a Banach space.

Proposition 4.1.5. *$(L^p(\Omega), ||\cdot||_p)$ is separable, that is, L^p contains countable dense subsets.*

Theorem 4.1.6 (Dominated convergence Theorem)**.** *[Roy63] Suppose that f_n is a sequence of measurable functions, that $f_n \longrightarrow f$ pointwise almost everywhere as $n \longrightarrow \infty$, and that $|f_n| \le g$ for all n, where g is integrable. Then f is integrable, and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

.

4.2 Sobolev Spaces

Definition 4.2.1 (Weak derivatives)**.** *Let $u \in L^1_{Loc}(\Omega)$ and $i \in \{1, \dots, \infty\}$ a function $v_i \in L^1_{Loc}$ is called weak i^{th} partial derivative of u if*

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega v_i \varphi dx \text{ for all } \varphi C^1_c(\Omega) \tag{4.8}$$

where $C_c^1(\Omega)$ denotes the class of functions continuous together with their first partial derivatives with compact support in Ω.

Notation: $vi = D_i u$

Set $Du = (D_1 u, \dots, D_p u)$. If $u \in C^1(\Omega)$ then $Du = \bigtriangledown u$ where $\bigtriangledown u$ is gradient of u.

Definition 4.2.2 (Sobolev space $W^{1,p}(\Omega)$). *We define $W^{1,p}(\Omega) := \{v \in L^p(\Omega) : Dv$ exist in $L^p(\Omega, \mathbb{R}^n)\}$ it is possible to show that the mapping*

$$v \in W^{1,p}(\Omega) \longrightarrow ||v||_{1,v} := ||v||_{L^p} + ||Dv||_{L^r}$$

is normed on $W^{1,p}(\Omega)$ and using the compactness of the L^p spaces and the uniqueness of the weak derivatives are shown that

$$\big(W^{1,p}(\Omega), ||\cdot||_{1,p}\big) \text{ is a Banach space.}$$

From separability of the L^p spaces $(1 \le p < \infty)$ follows also separability of $W^{1,p}(\Omega)$. On the other hand, from reflexivity of the L^p spaces $(1 \le p < \infty)$ follows also the $W^{1,p}(\Omega)$ is reflexive.

Proposition 4.2.3. *The space $C^1 \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ that is for every $u \in W^{1,p}(\Omega)$, there exists a sequence $\{u_n\}_n \subset C^1 \cap W^{1,p}(\Omega)$ such that $u_n \longrightarrow u$ and $\lim ||u_n - u||_{W^{1,p}(\Omega)} = 0$.*

Definition 4.2.4. *The closure of $C_c^1(\Omega)$ with respect to the norm $||\cdot||_{1,p}$ is called $W_0^{1,p}(\Omega)$, that is, $W_0^{1,p}(\Omega) = \overline{C_c^1(\Omega)}^{||\cdot||_{1,p}}$*

If $1 \le p < \infty$ then $W^{1,p}(\Omega)$ is separable as a closed subspace of separable space.

If $1 < p < \infty$ then $W^{1,p}(\Omega)$ is reflexive as a closed subspace of reflexive space.

Assumptions about domains: The boundary Γ of an open set $\Omega \subset \mathbb{R}^n$ is Lipschitz continuous if Γ is locally the graph of a Lipschitz continuous function. With a slight abuse of terminology, a domain with a Lipschitz boundary is also referred to as a Lipschitz domain. For more details about Lipschitz continuous domains we refer the reader to [Ada75].

In the following, we always assume that Ω is a Lipschitz domain, unless stated otherwise. $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}$; in other words, a function in $W^{1,p}(\Omega)$can be approximated by a sequence of functions smooth up to the boundary.

Trace Theorem: A uniformly continuous function v on a bounded domain Ω with boundary Γ has a well-defined boundary value, usually denoted by $v|\Gamma$. This property may be expressed in an alternative manner by the introduction of a map γ called the trace operator, which associates with each $v \in C(bar\Omega)$ its boundary value $\gamma v = v|\Gamma$, a function belonging to $C(\Gamma)$. For a function $v \in W^{1,p}(\Omega)$ the issue of its boundary value is less straightforward: the restriction of v to Γ need not make sense, since Γ is a set of measure zero, and two functions in $W^{1,p}(\Omega)$ are identified if they are equal a.e. Fortunately, it is possible to extend the notion of the trace operator for continuous functions in $C(\bar{\Omega})$ to functions in $W^{1,p}(\Omega)$ for certain ranges of the indices m and p. This result is summarized in the following.

Theorem 4.2.5 (Trace Theorem). *A There exists a unique bounded linear mapping $\gamma_0 : W^{1,p}(\Omega) \longrightarrow L^2(\Gamma)$ such that $\gamma_0 v = v|\Gamma$ when $v \in W^{1,p} \cap C(\bar{\Omega})$. The range of the mapping is $\gamma_0(W^{1,p}(\Omega)) = W^{\frac{1}{2},p}(\Gamma)$.*

The dual space of $W^{\frac{1}{2},p}(\Gamma)$denoted by $W^{-\frac{1}{2},p}(\Gamma)$.

For these fractional Sobolev spaces, we refer the interested reader to [Ada75]. In future, when the trace $\gamma_0 v$ of a Sobolev function v on the boundary is defined, we will simply write v for the trace $\gamma_0 v$.

Proposition 4.2.6.

$$W_0^{1,p}(\Omega) = \mathit{ker}(\gamma) = \{v \in W^{1,p}(\Omega) : \gamma(v) = 0\}$$

Theorem 4.2.7 (Embedding theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\Gamma = \partial\Omega$ be Lipschitz continuous,*

If $n > p$ then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, p^*]$ where $p^* = \frac{np}{n-p}$ with compact embedding when $q \in [1, p^*)$

If $n = p$ then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, \infty)$ where $p^* = \frac{np}{n-p}$.

If $n < p$ then $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ for every $q \in [1, \infty)$ where $p^* = \frac{np}{n-p}$ for all $0 \le \alpha \le 1 - \frac{1}{n}$ with compactness when $\alpha \in [0, 1 - \frac{p}{n}]$.

Theorem 4.2.8 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ open, bounded and connected, then there exists $C(\Omega) > 0$ such that $||u||_{L^p} \le || \bigtriangledown u||_{L^p}$ for all $u \in W^{1,p}(\Omega`)$*

From Poincaré inequality, it follows that the mapping

$$v \in W_0^{1,p}(\Omega) \longrightarrow || \bigtriangledown u||_{L^p} = | \bigtriangledown u|_{1,p}$$

is a norm in $W_0^{1,p}(\Omega)$ which is equivalent to the norm $|| \cdot ||_{1,p}$. Hence, we will be using either $|| \bigtriangledown u||_{L^p}$ or $| \bigtriangledown u|_{1,p}$ on $W^{1,p}(\Omega)$

Theorem 4.2.9 (Sobolev space as function space.). *[Zei85b] For each $k = 1, 2, 3, ...$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Proof. Let us show that $W^{k,p}(\Omega)$ is a norm.

1. $||u||_{W^{k,p}(\Omega)} \ge 0$ and $||u||_{W^{k,p}(\Omega)} = 0$ if $u = 0$ a.e.
2. $||\lambda u||_{W^{k,p}(\Omega)} = |\lambda| ||u||_{W^{k,p}(\Omega)}$
3. Let $u, v \in W^{k,p}(\Omega)$. If $1 \le p < \infty$, the Monkosiki's inequality implies

$$\begin{aligned} ||u+v||_{W^{k,p}(\Omega)} &= \sum \left(||D^\alpha u + D^\alpha v||^p_{L^p(\Omega)} \right)^{1/p} \\ &\le \left(\sum \left(||D^\alpha u||^p_{L^p(\Omega)} + ||D^\alpha v||^p_{L^p(\Omega)} \right)^p \right)^{1/p} \\ &\le \left(\sum ||D^\alpha u||^p_{L^p(\Omega)} \right)^{1/p} + \left(\sum ||D^\alpha v||^p_{L^p(\Omega)} \right)^{1/p} \\ &= ||u||_{W^{k,p}(\Omega)} + ||v||_{W^{k,p}(\Omega)} \end{aligned} \tag{4.9}$$

Hence, $W^{k,p}(\Omega)$ is a normed.

Now, we remain to show that $W^{k,p}(\Omega)$ is complete. Let us assume $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(\Omega)$. Then $\{D^\alpha u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^p(\Omega)$ for each $|\alpha| \le k$. Since $L^p(\Omega)$ is complete, there exist a function $u_\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_m \longrightarrow u_\alpha \quad \mathit{in}\ L^p(\Omega) \tag{4.10}$$

for each $|\alpha| \leq k$ in particular

$$u_m \longrightarrow u_{(0,0,\ldots,0)} =: u \quad \textit{in } L^p(\Omega)$$

Claim $\quad u \in W^{k,p}(\Omega)$ *and* $D^\alpha u = u_\alpha$

To verify this, let $\phi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\int_\Omega u D^\alpha \phi dx &= \int_\Omega \lim_{m\to\infty} u_m D^\alpha \phi dx \\ &= \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m \phi dx \\ &= \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega u_\alpha \phi dx.\end{aligned} \tag{4.11}$$

This implies the claim is valid. Since $D^\alpha u_m \longrightarrow D^\alpha u$ in $L^p(\Omega)$ for all $|\alpha| \leq k$, we have $u_m \longrightarrow u$ in $W^{k,p}(\Omega)$

Hence, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space. □

4.3 Elliptic Boundary Value Problem

Definition 4.3.1. *Let $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ (the boundary of Ω) be Lipschitz. then $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called Caratheodory function if, it satisfies [Rou00]:*

i. $a(\cdot, u, p)$ is measurable for every $u \in \mathbb{R}, p \in \mathbb{R}^2$.

ii. $a(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$.

Let a and d be to Caratheodory functions satisfying the growth conditions:

$$\begin{aligned}|a(x, r, s)| &\leq \gamma(x) + c|\gamma|^{\frac{p^*-\epsilon}{p'}} + c|s|^{p-1}, \quad \gamma \in L^{p'}(\Omega) \\ |c(x, r, s)| &\leq \beta(x) + c||^{p^*-\epsilon-1} + c|s|^{\frac{p}{p^{*\prime}}}, \quad \beta \in L^{p^{*\prime}}(\Omega).\end{aligned} \tag{4.12}$$

and let $f : \Omega \longrightarrow \mathbb{R}$ is measurable and consider the following boundary value problem and find $u : \Omega \longrightarrow \mathbb{R}$ such that [Rou00]

$$(P) \qquad \begin{cases} -\text{div } a(x, u(x), \bigtriangledown u(x)) + c(x, u(x), \bigtriangledown u(x)) = f(x) \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

To solve this we to introduce weak formulation of (P):

Let $v \in c\left(\overline{\Omega}\right)$ where $v = 0$ on $\partial\Omega$

$$-\sum_{i=1} \int_\Omega \frac{\partial a_i}{\partial x_i}(x, u(x), \bigtriangledown u(x)) v dx + \int_\Omega c(x, u(x), \bigtriangledown u(x)) v dx = \int f(x) v dx \tag{4.13}$$

implies

$$\int_\Omega a(x, u(x), \bigtriangledown u(x)) \cdot \bigtriangledown v dx + \int_\Omega c(x, u(x), \bigtriangledown u(x)) v dx = \int_\Omega f(x) v dx. \tag{4.14}$$

Then the weak formulation of the problem (P) of the function a and c satisfy growth conditions suggest that we can find the solution u in the space $W^{1,p}(\Omega)$. With $u = 0$ in $\partial\Omega, u \in W_0^{1,p}(\Omega)$ we obtain

$$(P_W) \qquad \begin{cases} \text{find } u \in W_0^{1,p}(\Omega) \text{ such that} \\ \int_\Omega a(x, u(x), \triangledown u(x)) \cdot \triangledown v dx + \int_\Omega c(x, u(x), \triangledown u(x)) v dx = \int_\Omega f(x) v dx, \text{ for all } v \in W_0^{1,p}(\Omega). \end{cases}$$

Now consider the operator $A : W_0^{1,p}(\Omega) \longrightarrow (W_0^{1,p}(\Omega))^* = W_0^{-1,p^*}(\Omega)$ defined by

$$\langle A(u), v\rangle = \int_\Omega a(x, u(x), \triangledown u(x) \cdot \triangledown v dx + \int_\Omega c(x, u(x), \triangledown u(x)) v dx. \tag{4.15}$$

Hence, (P_W) can be written as

$$(P_O) \qquad \begin{cases} \text{find } u \in W_0^{1,p}(\Omega) \text{ such that} \\ A(u) = f \end{cases}$$

To prove that (P_O) has a solution we have to show that the operator A satisfies the assumptions in Theorem 3.5.3, that are $f \in W_0^{1,p}$ and A is coercive and psuedomonotone.

Theorem 4.3.2. *Let $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ Lipschitz and suppose a, c of Caratheodory function satisfying the following conditions*

i.) There exist ϵ_1, ϵ_2 and there exist $k_1 \in L^1(\Omega)$ such that $a(x, r, s)s + c(x, r, s)r \geq \epsilon_1|s|^p + \epsilon_2|r|^q - k_1(x) \quad$ for $1 < q \leq p$

ii.) For almost all $x \in \Omega$ for all $r \in \mathbb{R}$ for all $s, \tilde{s} \in \mathbb{R}$ such that

$$(a(x, r, s) - a(x, r, \tilde{s}))(s - \tilde{s}) \geq 0 \tag{4.16}$$

then the problem P_W has a solution $u \in W_0^{1,p}(\Omega)$ provided $f \in L^{p^{*'}}$.

Proof. To prove that the operator A in P_O is coercive

$$\begin{aligned} \langle A(u), u\rangle &= \int_\Omega a(x, u(x), \triangledown u(x)) \cdot \triangledown u dx + \int_\Omega c(x, u(x), \triangledown u(x)) u dx \\ &\geq \epsilon_1 \int_\Omega |\triangledown u(x)|^p dx + \epsilon_2 \int_\Omega |u(x)|^q dx - ||k_1||_{L^1} \\ &\geq \min(\epsilon_1, \epsilon_2) \left[\int_\Omega |\triangledown u(x)|^p dx + \int_\Omega |u(x)|^q dx\right] - ||k_1||_{L^1} \\ &\geq \min(\epsilon_1, \epsilon_2)||u||^p W_0^{1,p}(\Omega) - ||k_1||_{L^1}(\Omega), \quad p > 1. \end{aligned} \tag{4.17}$$

This implies,

$$\frac{\langle A(u), u\rangle}{||u||W_0^{1,p}(\Omega)} \geq \min(\epsilon_1, \epsilon_2)||u||^{p-1}_{W_0^{1,p}(\Omega)} - \frac{||k_1||_{L^1}}{||u||_{W_0^{1,p}(\Omega)}}. \tag{4.18}$$

Hence,

$$\lim_{||u|| \to \infty} \frac{\langle A(u), u\rangle}{||u||_{W_0^{1,p}(\Omega)}} = \infty, \qquad \textit{since} \quad p > 1. \tag{4.19}$$

To prove A is bounded, consider

$$
\begin{aligned}
||A(u)||_{W_0^{1,p}(\Omega)} &= \sup \frac{\langle A(u), v\rangle}{||v||_{W_0^{1,p}(\Omega)}} \quad \textit{for } v \neq 0 \\
&\geq \frac{\langle A(u), v\rangle}{||v||_{W_0^{1,p}(\Omega)}} = \alpha \qquad (4.20) \\
|\alpha| = |\frac{\langle A(u), v\rangle}{||v||_{W_0^{1,p}(\Omega)}}| \geq 0.
\end{aligned}
$$

Now let's take

$$
\begin{aligned}
\langle A(u), v\rangle &= \int_\Omega a(x, u(x), \triangledown u(x)) \cdot \triangledown v dx + \int_\Omega c(x, u(x), \triangledown u(x)) v dx \\
&\leq \int_\Omega |a(x, u, \triangledown u)||\triangledown v| dx + \int_\Omega |c(x, u, \triangledown u)||v| dx \\
&\leq \int_\Omega \gamma(x)|\triangledown v| dx + c\int_\Omega |u(x)|^{\frac{p^*-\epsilon}{p'}}|\triangledown v(x)| dx + c\int_\Omega |\triangledown u|^{p-1}|\triangledown v(x)| dx \\
&+ \int_\Omega \beta(x)|v(x)| dx + \int_\Omega |u(x)|^{p^\epsilon - 1}|v(x)| dx + c\int_\Omega |\triangledown u(x)|^{\frac{p}{p^{*\prime}}}|v(x)| dx.
\end{aligned} \qquad (4.21)
$$

Consider each part of the expression (4.21):

$$
\begin{aligned}
&\int_\Omega \gamma(x)|\triangledown v| dx \leq ||\gamma(x)||_{L^{p'}}||\triangledown v||_{L^p} \\
&\int_\Omega |u(x)|^{\frac{p^*-\epsilon}{p'}}|\triangledown v(x)| dx \leq ||\triangledown v||_{L^p}\left(\int_\Omega |u(x)|^{p^*-\epsilon}\right)^{\frac{1}{p'}} \\
&\int_\Omega |\triangledown u|^{p-1}|\triangledown v(x)| dx \leq ||\triangledown v||_{L^p}||\triangledown u||_{l^p}^{\frac{p-1}{p}} \\
&\int_\Omega \beta(x)|v(x)| dx \leq ||v||_{L^{p^*}}||\beta||_{L^{p^{*\prime}}} \\
&\int_\Omega |u(x)|^{p^\epsilon - 1}|v(x)| dx \leq \left(\int_\Omega |v|^s dx\right)^{\frac{1}{s}}\left(\int_\Omega (|u(x)|^{p^*-\epsilon-1})s'\right)^{\frac{1}{s'}} \quad \textit{for} \quad s = p^* - \epsilon \\
&\int_\Omega |\triangledown u(x)|^{\frac{p}{p^{*\prime}}}|v(x)| dx \leq \int_\Omega |\triangledown u(x)|^{\frac{p}{p^{*\prime}}}|v(x)| dx.
\end{aligned} \qquad (4.22)
$$

By taking the total summation the inequality (4.22)

$$
\langle A(u), v\rangle \leq ||\triangledown v||_{L^p}\left[||\gamma||_{L^{p'}} + ||u||_{L^{p^*-\epsilon}}^{\frac{p^*-\epsilon}{p'}} + ||\triangledown u||_{L^p}^{\frac{p-1}{p}} + c||\beta||_{L^{p^{*\prime}}} + c||u||_{L^{p^*-\epsilon}}^{p^*-\epsilon-1}\right] \qquad (4.23)
$$

implies

$$
||A(u)||_{W_0^{1,p}(\Omega)} \leq \left[||\gamma||_{L^{p'}} + ||u||_{L^{p^*-\epsilon}}^{\frac{p^*-\epsilon}{p'}} + ||\triangledown u||_{L^p}^{\frac{p-1}{p}} + c||\beta||_{L^{p^{*\prime}}} + c||u||_{L^{p^*-\epsilon}}^{p^*-\epsilon-1}\right] = M \qquad (4.24)
$$

for some $M \in \mathbb{R}$. Hence, A is bounded.

To show A is pseudomonotone, we already know that A is bounded. Let $u_n \rightharpoonup u$ in $W_0^{1,p}$ be such that

$$
\liminf_{n\to\infty} \langle A(u), u_n - u\rangle \leq 0. \qquad (4.25)
$$

Now let us prove that $\liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \geq \langle A(u), u - v\rangle$.

$$\langle A(u_n), u_n - v\rangle = \int_\Omega a(x, u_n, \triangledown u_n)\cdot\triangledown(u_n - v)dx + \int_\Omega c(x, u_n, \triangledown u_n)\cdot\triangledown(u_n - v)dx. \tag{4.26}$$

Consider $B : W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega) \longrightarrow (W_0^{1,p}(\Omega))'$ where $\langle B(w,u), v\rangle = \int_\Omega a(x, w, \triangledown u)\cdot\triangledown(v)dx + \int_\Omega c(x, w, \triangledown w)vdx$ such that $B(u,u) = A(u)$,for $\epsilon\in[0,1], u_\epsilon = \epsilon v + (1-\epsilon)u$

Let us show that

$$\langle B(u_n,u_n) - B(u_n,u_\epsilon), u_n - u_\epsilon\rangle \geq 0 \tag{4.27}$$

$$\int_\Omega [a(x,u_n,\triangledown u_n) - a(x,u_n,\triangledown u_\epsilon)][\triangledown u_n - \triangledown u_\epsilon]\,dx + \int_\Omega [c(x,u_n,\triangledown u_n) - c(x,u_n,\triangledown u_n)][u_n - u_\epsilon]\,dx \tag{4.28}$$

and by the inequality (4.16)

$$\int_\Omega [a(x,u_n,\triangledown u_n) - a(x,u_n,\triangledown u_\epsilon)][\triangledown u_n - \triangledown u_\epsilon]\,dx \geq 0. \tag{4.29}$$

Hence, (4.27) holds and which implies

$$\langle A(u_n), u_n - u_\epsilon\rangle \geq \langle B(u_n,u_\epsilon), u_n - u_\epsilon\rangle \tag{4.30}$$

Here, $u_n - u_\epsilon = u_n - \epsilon v - (1-\epsilon)u = u_n + \epsilon(u - v)$ which implies

$$\langle A(u_n), u - v\rangle + \epsilon\langle A(u_n), u - v\rangle \geq \langle B(u_n,u_\epsilon), u_n - u\rangle + \epsilon\langle B(u_n,u_\epsilon), u - v\rangle \tag{4.31}$$

$$\epsilon\langle A(u_n), u - v\rangle \geq -\langle A(u_n), u - v\rangle\langle B(u_n,u_\epsilon), u_n - u\rangle + \epsilon\langle +B(u_n,u_\epsilon), u - v\rangle \tag{4.32}$$

Assume that we have proved

$$\lim_{n\to\infty}\langle B(u_n,v), u_n - u\rangle = 0 \tag{4.33}$$

and

$$\lim_{n\to\infty}\langle B(u_n,v), w\rangle = \langle B(u,v), w\rangle \quad \textit{for all } w, v\in W_0^{1,p}(\Omega) \tag{4.34}$$

By taking $v = u_\epsilon$ implies

$$\lim_{n\to\infty}\langle B(u_n,u_\epsilon), u_n - u\rangle = 0 \tag{4.35}$$

$$\lim_{n\to\infty}\langle B(u_n,u_\epsilon), w\rangle = \langle B(u,u_\epsilon), w\rangle \quad \textit{for} \quad w = u - v \quad \textit{from } (4.33) \tag{4.36}$$

$$\lim_{n\to\infty}\langle B(u_n,u_\epsilon), u - v\rangle = \langle B(u,u_\epsilon), u - v\rangle \tag{4.37}$$

and pass to the limit in (4.32) when $u \longrightarrow \infty$

$$\begin{aligned}\epsilon\liminf_{n\to\infty}\epsilon\langle A(u_n), u - v\rangle &\geq -\limsup_{n\to\infty}\langle A(u_n), u - v\rangle\langle\lim_{n\to\infty} B(u_n,u_\epsilon), u_n - u\rangle + \epsilon\lim_{n\to\infty}\langle +B(u_n,u_\epsilon), u - v\rangle\\ &\geq \langle B(u,u_\epsilon), u - v\rangle\end{aligned} \tag{4.38}$$

$$\lim_{n\to\infty}\langle B(u,u_\epsilon), u - v\rangle = \langle B(u,u), u - v\rangle \tag{4.39}$$

$$\liminf_{n\to\infty}\langle A(u_n), u - v\rangle \geq \langle A(u), u - v\rangle \tag{4.40}$$

by using the fact in (4.27) we have

$$\begin{aligned}\langle A(u_n), u_n - v\rangle &= \langle A(u_n), u_n - u\rangle + \langle A(u_n), u - v\rangle \\ &= \langle B(u_n, u_n), u_n - u\rangle + \langle A(u_n), u - v\rangle \\ &= \langle B(u_n, u), u_n - u\rangle + \langle B(u_n, u_n) - B(u_n, u), u_n - v\rangle + \langle A(u_n), u - v\rangle\end{aligned} \tag{4.41}$$

implies

$$\begin{aligned}\liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle =& \liminf_{n\to\infty}\langle B(u_n, u), u_n - u\rangle + \liminf_{n\to\infty}\langle B(u_n, u_n) - B(u_n, u), u_n - v\rangle \\ &+ \liminf_{n\to\infty}\langle A(u_n), u - v\rangle\end{aligned} \tag{4.42}$$

from the inequalities (4.27) and (4.33) we have

$$\liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \geq \liminf_{n\to\infty}\langle A(u_n), u - v\rangle \geq \langle A(u_n), u - v\rangle. \tag{4.43}$$

Now we remain to prove (4.33) and (4.34).

Since $u_n \longrightarrow n$ in $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < p^*, q = p^* = \epsilon$ implies $u_n \longrightarrow u$ in $L^{p^*-\epsilon}$. By definition

$$\langle B(u_n, v), w\rangle = \int_\Omega a(x, u_n, \bigtriangledown u_n) \bigtriangledown w dx + \int_\Omega c(x, u_n, \bigtriangledown u_n) w dx \tag{4.44}$$

Since $a(x, u_n, \bigtriangledown v) \longrightarrow a(x, u, \bigtriangledown v)$ strongly in $L^{p'}(\Omega)$ there exist

$$\{n_k\} \textit{ such that } \{u_{u_k}(x)\} \longrightarrow u(x) \quad \textit{a.e.} \tag{4.45}$$

and

$$g \in L^{p^*-\epsilon} \textit{ such that } |u_{n_k}(x)| < g(x) \textit{ a.e. }. \tag{4.46}$$

We have

$$\begin{aligned}&a(x, u_{u_k}(x), \bigtriangledown v(x)) \longrightarrow a(x, u(x), \bigtriangledown v(x)) \textit{ a.e.} \\ &|a(x, u_{u_k}(x), \bigtriangledown v(x)) - a(x, u(x), \bigtriangledown v(x))|^{p'} \longrightarrow 0 \textit{ a.e.}\end{aligned} \tag{4.47}$$

implies

$$\int_\Omega |a(x, u_{u_k}(x), \bigtriangledown v(x)) - a(x, u(x), \bigtriangledown v(x))|^{p'} dx = 0 \textit{ in } L^{p'}. \tag{4.48}$$

To show (4.48)

$$\begin{aligned}&|a(x, u_{u_k}(x), \bigtriangledown v(x)) - a(x, u(x), \bigtriangledown v(x))|^{p'} \\ &\leq 2^{p'}\left[|a(x, u_{u_k}(x), \bigtriangledown v(x))|^{p'} + |a(x, u(x), \bigtriangledown v(x))|^{p'}\right] \\ &= 2^{p'}\left[|\gamma(x)| + c|u_{n_k}(x)|^{\frac{p^*-\epsilon}{p'}} + c|\bigtriangledown v(x)|^{p'-1} + |\gamma| + c|u_{n_k}(x)| + c|\bigtriangledown v(x)|^{p'-1}\right] \qquad \textit{(by 4.12)} \\ &\leq M\left[|\gamma(x)|^{p'} + |u_{n_k}(x)|^{p^*-\epsilon} + |u(x)|^{p^*-\epsilon} + |\bigtriangledown v|^p\right] \\ &\leq M\left[|\gamma(x)|^{p'} + |g(x)|^{p^*-\epsilon} + |u(x)|^{p^*-\epsilon} + |\bigtriangledown v|^p\right] \quad (\textit{ by } 4.46).\end{aligned} \tag{4.49}$$

Hence, (4.48) is in L^1. By the dominated convergence theorem

$$\lim_{n\to\infty}\int_\Omega |a(x,u_n(x),\bigtriangledown v(x))-a(x,u(x),\bigtriangledown v(x))|^{p'}dx=0 \tag{4.50}$$

implies

$$\lim_{n\to\infty}\int_\Omega a(x,u_n(x),\bigtriangledown v(x))wdx=\lim_{n\to\infty}\int_\Omega a(x,u(x),\bigtriangledown v(x))wdx \tag{4.51}$$

and

$$\lim_{n\to\infty}\int_\Omega a(x,u_n(x),\bigtriangledown v(x))\bigtriangledown(u_n-u)dx=0. \tag{4.52}$$

Now let us prove the c part.

Case I

Let c be a function of x and r . Since $c(x,u_n)\longrightarrow c(x,u)$ in $L^{p^{*\prime}}$

$$\lim_{n\to\infty}\int_\Omega |c(x,u_n(x))-c(x,u(x))|^{p^{*\prime}}dx=0 \tag{4.53}$$

since $\{u_n\}\longrightarrow u$ in $L^{p^*-\epsilon}$ there exist $\{n_k\}$ *such that* $\{u_{n_k}(x)\}\longrightarrow u(x)$ *a.e.* and $g\in L^{p^*-\epsilon}$ such that $|u_{n_k}(x)|<g(x)$ *a.e.*

then $c(x,u_{n_k}(x))\longrightarrow c(x,u(x))$ a.e. and hence,

$$\begin{aligned}|c(x,u_{n_k}(x))-c(x,u(x))|^{p^{*\prime}} &\le 2^{p^{*\prime}}\left(|c(x,u_{n_k}(x))|^{p^{*\prime}}+|c(x,u(x))|^{p^{*\prime}}\right)\\ &\le K\left[|\gamma(x)|^{p^{*\prime}}+\left(|u_{n_k}(x)|^{p^*-\epsilon-1}\right)^{p^{*\prime}}+\left(|u(x)|^{p^*-\epsilon-1}\right)^{p^{*\prime}}\right]\\ &\le K\left[|\gamma(x)|^{p^{*\prime}}+\left(g(x)|^{p^*-\epsilon-1}\right)^{p^{*\prime}}\right] \qquad \textit{in } L^1(\Omega)\end{aligned} \tag{4.54}$$

Let $\alpha=(p^{*-\epsilon-1})\frac{p^*}{p^*-1}<p^*-\epsilon$ implies $1<\frac{p^*-\epsilon}{\alpha}$. Hence,

$$\left(g(x)|^{p^*-\epsilon-1}\right)^{p^{*\prime}}=g(x)^{(p^{*-\epsilon-1})\frac{p^*}{p^*-1}} \tag{4.55}$$

$$\left(\int_\Omega\left[g(x)^{(p^{*-\epsilon-1})\frac{p^*}{p^*-1}}dx\right]^{\frac{\alpha}{p^*-\epsilon}}\right)\left(\int_\Omega 1^{\frac{\frac{\alpha}{p^*-\epsilon}}{\frac{\alpha}{p^*-\epsilon}-1}}\right)^{\frac{\frac{\alpha}{p^*-\epsilon}-1}{\frac{\alpha}{p^*-\epsilon}}} \tag{4.56}$$

$$\left(\int_\Omega |g(x)|^{\frac{\alpha}{p^*-\epsilon}}dx\right)^{\frac{\alpha}{p^*-\epsilon}}|\Omega|^{\frac{\alpha-p^*+\epsilon}{\alpha}}<\infty. \tag{4.57}$$

By the dominated convergence theorem

$$\lim_{k\to\infty}\int_\Omega |c(x,u_{n_k}(x))-c(x,u(x))|^{p^{*\prime}}dx=0 \tag{4.58}$$

and $c(x,u_n(x))\longrightarrow c(x,u(x))|^{p^{*\prime}}$ strongly in $L^{p^*}(\Omega)$. Let $w\in W_0^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ imply that $w\in L^{p^*}(\Omega)$ then we have

$$\lim_{n\to\infty}\int_\Omega c(x,u_n(x))wdx=\int_\Omega c(x,u(x))wdx, \tag{4.59}$$

and since $c(x, u_n(x)) \longrightarrow c(x, u)$ in $L^{*'}(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega c(x, u_n(x),)(u_n(x) - u)dx = 0. \tag{4.60}$$

From equations (4.51), (4.52), (4.59) and (4.60) the equation case I of (4.33) and(4.34) holds.

Case II

Assume that $c(x, r, s) = \bar{c}(x, r).s$ where $\bar{c} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^n$.

i. $\bar{c}(x, \cdot)$ is measurable for all $\mathbb{R}$.

ii. $\bar{c}(x, \cdot)$ is continues a.e. in $x \in \Omega$.

Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_n \longrightarrow u$ in $L^{*-\epsilon}(\Omega)$ and $\bigtriangledown u_n \longrightarrow \bigtriangledown u$ in $L^p(\Omega)$. Moreover $|| \bigtriangledown u_n||_{L^p}$ is bounded.

$u_n \longrightarrow u$ implies that $\bar{c}(x, u_n) \longrightarrow \bar{c}(x, u)$ strongly in $L^{q+\epsilon_1}(\Omega)$ which implies $||\bar{c}(x, u_n)||_{L^{q+\epsilon_1}}$ is bounded and by applying Theorem 4.1.3,

$$\lim_{n\to\infty} \int_\Omega \bar{c}(x, u_n) \cdot \bigtriangledown u_n(u_n - u)dx \le c \left(\int_\Omega |\bar{c}(x, u_n) \bigtriangledown u_n(u_n - u)|^\alpha \right)^{\frac{1}{\alpha}} \tag{4.61}$$

$$||\bar{c}(x, u_n)||_{L^{q+\epsilon_1}} || \bigtriangledown u_n||_{L^p} ||u_n - u||_{L^{p^*-\epsilon}} \longrightarrow 0. \tag{4.62}$$

Hence,

$$\lim_{n\to\infty} \int_\Omega \bar{c}(x, u_n) \cdot \bigtriangledown u_n(u_n - u)dx = 0. \tag{4.63}$$

From **Case I** and **Case II** the c part of (4.33) and (4.34) holds and by taking the combination with (4.52) completes the proof of (4.33) and (4.34) .

Hence, by (4.19) and (4.43) the problem P has at least one solution. □

5. Conclusion

In this essay, we explored a well known method of solving Dirichlet boundary value problems governed by non-linear second order elliptical equations. The method called energy method consists in introducing a weak formulation of the boundary value problem that one written also in the form of an abstract equation involving an operator which is coercive and psuedomonotone, usually then the weak formulation having a solution.

This project allowed me to learn about the functional spaces like L^p space, Sobolev space and their properties relevant to the project, the monotone and psuedomonotone operators.

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