Algebraic Geometry and Computer Algebra

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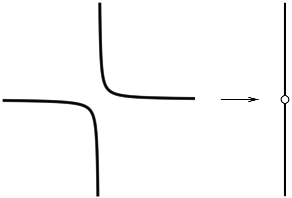
Noether Normalization and Dimension

We have seen that the image of an algebraic set under a projection map needs not be an algebraic set.



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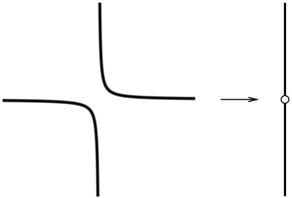

Under an additional hypothesis, however, projections are better behaved.

Noether Normalization and Dimension

Projection Theorem. Let $I \subset K[x_1, \ldots, x_n]$ be an ideal, and let $I_1 = I \cap K[x_2, \ldots, x_n]$ be its first elimination ideal. Suppose that I contains a polynomial f which is monic in x_1 of some degree $d \geq 1$:

$$f = x_1^d + c_1(x_2, \ldots, x_n)x_1^{d-1} + \ldots + c_d(x_2, \ldots, x_n),$$

with coefficients $c_i \in K[x_2, \ldots, x_n]$. Let

$$\pi_1 : \mathbb{A}^n(\overline{K}) \to \mathbb{A}^{n-1}(\overline{K}),\ (a_1, \ldots, a_n) \mapsto (a_2, \ldots, a_n),$$

be projection onto the last $n-1$ components, and let $A = \mathrm{V}(I) \subset \mathbb{A}^n(\overline{K})$. Then

$$\pi_1(A) = \mathrm{V}(I_1) \subset \mathbb{A}^{n-1}(\overline{K}).$$

In particular, $\pi_1(A)$ is an algebraic set.

Noether Normalization and Dimension

The extra hypothesis of the projection theorem can always be achieved by a change of coordinates:

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Lemma. Let K be an infinite field. Let $a_2, \ldots, a_n \in K$ be sufficiently general. Substituting

$$x_i = \widetilde{x}_i + a_i x_1$$

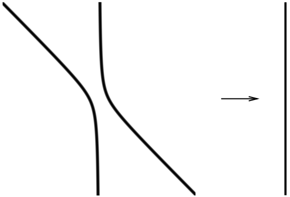
in f, $i = 2, \ldots, n$, we get a polynomial of type

$$a x_1^d + c_1(\widetilde{x}_2, \ldots, \widetilde{x}_n) x_1^{d-1} + \ldots + c_d(\widetilde{x}_2, \ldots, \widetilde{x}_n),$$

where $a \in K$ is a nonzero scalar, $d \geq 1$, and each $c_i \in K[\widetilde{x}_2, \ldots, \widetilde{x}_n]$.

Noether Normalization and Dimension

Example. Substituting $y = \widetilde{y} + x$ in $xy - 1$, we get the polynomial $x^2 + x\widetilde{y} - 1$ which is monic in x. Accordingly, the hyperbola $C = \mathrm{V}(xy - 1)$ projects *onto* $\mathbb{A}^1(\mathbb{R})$ via $(a, b) \mapsto (a, b - a) \mapsto b - a$:



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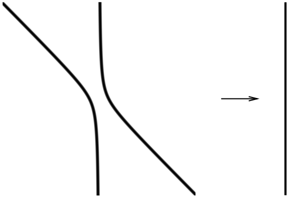


The projection theorem can be used to prove the weak version of the Nullstellensatz.

Proof of the Nullstellensatz, Weak Version. If $I \subset K[x_1, \dots, x_n]$ is an ideal containing 1, its locus of zeros in $\mathbb{A}^n(\overline{K})$ is empty.

For the converse, suppose that the result is true for polynomials in $n-1$ variables, and let $I \subset K[x_1, \dots, x_n]$ be an ideal such that $1 \notin I$. We have to show that $\mathrm{V}(I) \subset \mathbb{A}^n(\overline{K})$ is nonempty. This is clear if $I = \langle 0 \rangle$. If I is nonzero, pick a nonconstant polynomial $f \in I$. In suitable coordinates $x_1, \widetilde{x}_2, \dots \widetilde{x}_n$, chosen as in the lemma, f becomes a monic polynomial in x_1 as required by the extra hypothesis of the projection theorem (adjust the constant leading term in x_1, if necessary). Since $1 \notin I$, we have $1 \notin I \cap K[\widetilde{x}_2, \dots, \widetilde{x}_n]$ as well. It follows from the induction hypothesis that $\mathrm{V}(I \cap K[\widetilde{x}_2, \dots, \widetilde{x}_n]) \subset \mathbb{A}^{n-1}(\overline{K})$ contains a point. By the projection theorem, this point is the image of a point in $\mathrm{V}(I)$ under the projection which maps $(a_1, a_2, \dots, a_n)$ to $(\widetilde{a}_2, \dots, \widetilde{a}_n)$. In particular, $\mathrm{V}(I)$ is nonempty, and we are done by induction

Remark. Let $\langle 0\rangle \subsetneq I \subsetneq K[x_1,\dots,x_n]$ be an ideal.

(i) Successively carrying out the induction step in the proof above, applying the lemma at each stage, we may suppose that the coordinates are chosen such that *each* nonzero elimination ideal $I_{k-1} = I \cap K[x_k, x_{k+1},\dots,x_n]$ contains a monic polynomial of type

$$\begin{aligned} f_k &= x_k^{d_k} + c_1^{(k)}(x_{k+1},\dots,x_n)x_k^{d_k-1} + \ldots + c_{d_k}^{(k)}(x_{k+1},\dots,x_n) \\ &\in K[x_{k+1},\dots,x_n][x_k]. \end{aligned}$$

Then, if $1 \le c \le n$ is minimal with $I_c = \langle 0\rangle$, each projection map

$$\pi_k : \mathrm{V}(I_{k-1}) \to \mathrm{V}(I_k),\ (a_k, a_{k+1},\dots,a_n) \mapsto (a_{k+1},\dots,a_n),$$

$1 \le k \le c$, is surjective.

Hence, the composite map

$$\pi = \pi_c \circ \cdots \circ \pi_1 : \mathrm{V}(I) \to \mathbb{A}^{n-c}(\overline{K}).$$

is surjective as well. Furthermore, the π_k and, thus, π have finite fibers: if a point $(a_{k+1}, \ldots, a_n) \in \mathrm{V}(I_k)$ can be extended to a point $(a_k, a_{k+1}, \ldots, a_n) \in \mathrm{V}(I_{k-1})$, then a_k must be among the finitely many roots of the univariate polynomial $f_k(x_k, a_{k+1}, \ldots, a_n) \in \overline{K}[x_k]$.

Noether Normalization and Dimension

In practical terms, combining the above with univariate root finding, we get a recipe for finding explicit points of $\mathrm{V}(I)$.

(A) Compute a lexicographic Gröbner basis $\mathcal{G}$ for I. Then $\mathcal{G}$ contains lexicographic Gröbner bases for the whole flag of elimination ideals I_{k-1}, $k = 1, \ldots, n$. Moreover, the extra hypothesis of the projection theorem is fulfilled for *each* $I_{k-1} \neq \langle 0 \rangle$ iff polynomials f_k as in (i) are among the Gröbner basis elements (up to nonzero scalar factors).

(B) In this case, every point $(a_{c+1}, \ldots, a_n) \in \mathbb{A}^{n-c}(\overline{K})$ can be extended to a point $(a_1, \ldots, a_c, a_{c+1}, \ldots, a_n) \in \mathrm{V}(I)$ by building up one coordinate at a time:

Noether Normalization and Dimension

If $(a_{k+1}, \dots, a_{c+1}, \dots, a_n) \in \mathrm{V}(I_k) \subset \mathbb{A}^{n-k}(\overline{K})$ has already been chosen, consider the map

$$\Phi_k \colon K[x_k, x_{k+1}, \dots, x_n] \to \overline{K}[x_k],\ x_{k+1} \mapsto a_{k+1}, \dots, x_n \mapsto a_n.$$

The image $\Phi_k(I_{k-1})$ is a principal ideal generated by the greatest common divisor of the images of the elements of $\mathcal{G} \cap K[x_k, x_{k+1}, \dots, x_n]$. Pick a_k to be a root of that generator.

(C) If one monic polynomial is missing, start over again in new coordinates.

Elimination Orders

Singular Example.

```
ring R = 0, (x,y,z), lp;
poly f1 = y3z-2y2z-z3+x2+z;
poly f2 = xy3z-2xy2z-xz3+x3+y3-2y2+xz-z2+y;
ideal I = f1, f2;
option(redSB);
ideal SI = groebner(I);
SI;
SI[1]=y3-2y2+y-z2
SI[2]=x2-yz+z
```

Noether Normalization and Dimension

Example. Consider the curve $C = \mathrm{V}(f_1, f_2) \subset \mathbb{A}^3(\mathbb{C})$, where

$$\begin{aligned} f_1 &= y^3z - 2y^2z - z^3 + x^2 + z, \\ f_2 &= xy^3z - 2xy^2z - xz^3 + x^3 + y^3 - 2y^2 + xz - z^2 + y. \end{aligned}$$

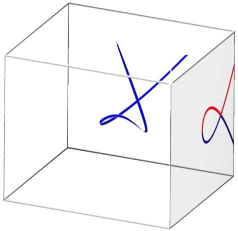
Computing the reduced lexicographic Gröbner basis for the ideal $\langle f_1, f_2 \rangle$, we get the two polynomials below:

$$x^2 - yz + z, \qquad y^3 - 2y^2 + y - z^2.$$

The first Gröbner basis element is monic in x of degree 2. Thus, projection of C to the yz-plane is $2:1$ and *onto* the curve C_1 defined by the second Gröbner basis element. In turn, C_1 is projected $3:1$ *onto* the z-axis. In sum, C is projected $6:1$ onto the z-axis.

Noether Normalization and Dimension

The real picture below shows both curves C and C_1. Only the blue part of C_1 has real preimage points on C. The red part has complex preimage points.



Noether Normalization and Dimension

Given an ideal $\langle 0 \rangle \subsetneq I \subsetneq K[x_1, \ldots, x_n]$, the proof of the Nullstellensatz yields a composition of projections

$$\pi = \pi_c \circ \cdots \circ \pi_1 : A = \mathrm{V}(I) \to \mathbb{A}^{n-c}$$

which is surjective and has finite fibers:

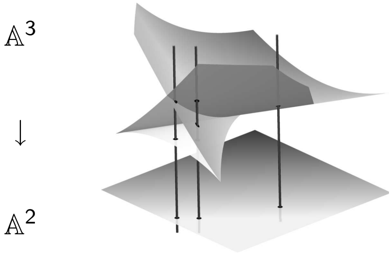


Figure: *We project a surface which is called the* **swallowtail**.

Noether Normalization and Dimension

Intuitively, the number $d = n - c$ should be the dimension of A. To make this a formal definition, it is convenient to work on the level of rings.

Noether Normalization and Dimension

In the situation of the projection theorem, if $\pi_1 : \mathrm{V}(I) \to \mathrm{V}(I_1)$ is projection onto the last $n-1$ components, the extra hypothesis of the theorem guarantees that π_1 is surjective with finite fibers.

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To study the ring theoretic analog of π_1, we introduce the following notation:

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To study the ring theoretic analog of π_1, we introduce the following notation:

If R is a subring of a ring S, we say that $R \subset S$ is a **ring extension**. More generally, if $R \to S$ is any injective ring homomorphism, we identify R with its image in S and consider, thus, $R \subset S$ as a ring extension.

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With this notation, the algebraic counterpart of the map π_1 is the ring extension

$$R = K[x_2, \ldots, x_n]/I_1 \subset S = K[x_1, \ldots, x_n]/I$$

which is induced by the inclusion $K[x_2, , \ldots, x_n] \subset K[x_1, \ldots, x_n]$.

Noether Normalization and Dimension

We may, then, rephrase the extra hypothesis of the projection theorem by saying that the element $\overline{x}_1 = x_1 + I \in S$ is integral over R in the following sense:

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Definition. Let $R \subset S$ be a ring extension. An element $s \in S$ is said to be **integral over R** if it satisfies a monic polynomial equation

$$s^d + r_1 s^{d-1} + \ldots + r_d = 0, \text{ with all } r_i \in R.$$

The equation is, then, called an **integral equation** for s over R. If every element $s \in S$ is integral over R, we say that S is **integral over R**, or that $R \subset S$ is an **integral extension**.

Integrality Criterion. Let I be an ideal of $K[x_1, \dots, x_n]$, and let $\overline{f}_1 = f_1 + I, \dots, \overline{f}_m = f_m + I \in K[x_1, \dots, x_n]/I$. Consider a polynomial ring $K[y_1, \dots, y_m]$, the homomorphism

$$\phi : K[y_1, \dots, y_m] \to S = K[x_1, \dots, x_n]/I, \; y_i \mapsto \overline{f}_i,$$

and the ideal

$$J = I\, K[\mathbf{x}, \mathbf{y}] + \langle f_1 - y_1, \dots, f_m - y_m \rangle \subset K[\mathbf{x}, \mathbf{y}].$$

Let $>$ be an elimination order on $K[\mathbf{x}, \mathbf{y}]$ with respect to $x_1, \dots, x_n$, and let $\mathcal{G}$ be a Gröbner basis for J with respect to $>$. Then we already know that the elements of $\mathcal{G} \cap K[\mathbf{y}]$ generate $\ker \phi$. View $R := K[y_1, \dots, y_m]/\ker \phi$ as a subring of S by means of ϕ. Show that $R \subset S$ is integral iff for each i, $1 \leq i \leq n$, there is an element of $\mathcal{G}$ whose leading monomial is of type $x_i^{\alpha_i}$ for some $\alpha_i \geq 1$.

Noether Normalization and Dimension

Example. Consider the homomorphism of polynomial rings

$$\phi : K[x,y,z] \to K[s,t],\ x \mapsto s,\ y \mapsto t^2-1,\ z \mapsto t(t^2-1).$$

Computing the reduced lexicographic Gröbner basis for the ideal

$$J = \langle s-x,\ t^2-1-y,\ t(t^2-1)-z \rangle,$$

we get the polynomials

$$\begin{array}{lll} y^3+y^2-z^2, & tz-y^2-y, & ty-z, \\ t^2-y-1, & s-x. & \end{array}$$

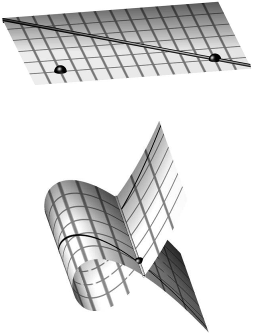
Inspecting the Gröbner basis elements, we find: The kernel of ϕ is the principal ideal generated by the first Gröbner basis element $z^2-y^2(y+1)$, and the induced ring extension

$$R = K[x,y,z]/\langle z^2-y^2(y+1)\rangle \subset S = K[s,t]$$

is integral.

Noether Normalization and Dimension

Geometrically, the map $\mathbb{A}^2 \to \mathbb{A}^3$ corresponding to the ring extension parametrizes $\mathrm{V}(z^2 - y^2(y+1))$:



Singular Example.

```
> ring RR = 0, (s,t,x,y,z), lp;
> ideal J = s-x, t2-1-y, t3-t-z;
> option(redSB);
> ideal SJ = groebner(J);
> SJ;
SJ[1]=y3+y2-z2
SJ[2]=tz-y2-y
SJ[3]=ty-z
SJ[4]=t2-y-1
SJ[5]=s-x
```

The Singular command for testing this is `finitenessTest`.

Now, we can formulate the ring theoretic analog to the proof of the Nullstellensatz.

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Theorem. Given a quotient ring $S = K[x_1, \ldots, x_n]/I$, there are elements $y_1, \ldots, y_d \in S$ such that:

(i) $y_1, \ldots, y_d$ are algebraically independent over K.

(ii) $K[y_1, \ldots, y_d] \subset S$ is an integral ring extension.

If $y_1, \ldots, y_d$ satisfy conditions (i) and (ii), the inclusion

$$K[y_1, \ldots, y_d] \subset S$$

is called a **Noether normalization** for S.

Noether Normalization and Dimension

Our remark following the proof of the Nullstellensatz shows one way of computing a Noether normalization: combine lexicographic Gröbner basis computations with randomly chosen coordinate transformations.

Singular Example.

```
> ring R = 0, (x,y), dp;
> ideal I = xy-1;
> LIB"algebra.lib";
> noetherNormal(I);
[1]:
   _[1]=x
   _[2]=4x+y
[2]:
   _[1]=y
```

Noether Normalization and Dimension

Definition. Let $I \subset K[x_1, \ldots, x_n]$ be an ideal and let $A = \mathrm{V}(I) \subset \mathbb{A}^n(\overline{K})$ be its vanishing locus over the algebraic closure of K. If $A \neq \emptyset$, and

$$K[y_1, \ldots, y_d] \subset K[x_1, \ldots, x_n]/I$$

is a Noether normalization, we define d to be the **dimension** of A, written

$$\dim A = d.$$

By convention, the dimension of the empty subset of $\mathbb{A}^n(\overline{K})$ is -1.

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Theorem. The definition is independent of all choices made.

Noether Normalization and Dimension

In principle, we can compute the dimension by computing a Noether normalization.

Singular Example. Twisted cubic curve:

```
> ring R = 0, (x,y,z), dp;
> ideal I = y-x2, z-x3;
> LIB"algebra.lib";
> noetherNormal(I);
[1]:
   _[1]=x
   _[2]=9x+y
   _[3]=3x+y+z
[2]:
   _[1]=z
```

Singular Example. Whitney umbrella:

```
> ring R = 0, (x,y,z), dp;
> ideal I = x2-y2z;
> LIB"algebra.lib";
> noetherNormal(I);
[1]:
   _[1]=x
   _[2]=3x+y
   _[3]=9x+3y+z
[2]:
   _[1]=y
   _[2]=z
```

Noether Normalization and Dimension

Combining lexicographic Gröbner basis computations with randomly chosen coordinate transformations can be very slow. If one is just interested to compute the dimension, the following result gives a much more efficient algorithm:

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Theorem. Let $I \subsetneq K[x_1, \dots, x_n]$ be an ideal, let $\mathrm{V}(I)$ be its locus of zeros in $\mathbb{A}^n(\overline{K})$, and let $>$ be a global monomial order on $K[x_1, \dots, x_n]$. Then

$$\dim \mathrm{V}(I) = d,$$

where d is the maximum cardinality of a subset of variables $\mathbf{u} \subset \{x_1, \dots, x_n\}$ such that

$$\mathbf{L}(I) \cap K[\mathbf{u}] = \langle 0 \rangle.$$

Noether Normalization and Dimension

Singular Example. Considering the reduced Gröbner basis

$$f_1 = x^2 - y, \quad f_2 = xy - z, \quad f_3 = y^2 - xz$$

for the ideal $\mathrm{I}(C)$ of the twisted cubic curve with respect to dp, we find that $\mathbf{u} = \{z\}$ is a set of variables of maximal cardinality such that

$$\langle x^2, xy, y^2 \rangle \cap K[\mathbf{u}] = \langle 0 \rangle.$$

This shows once more that the dimension of C is 1.

Singular Example. Twisted cubic curve:

```
> ring R = 0, (x,y,z), dp;
> ideal I = y-x2, z-x3;
> dim(groebner(I));
1
```