Problem solving in physics: part 2

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0. Problems in physics

- Why are things the way they are?
- What happens next?
- How do we get there from here?

How to solve a problem

1. Write down the problem.
2. Write down the answer.
3. Find a way to get from 1 to 2.. . .

(i.e. Build a model. . .)

Problem solving

From the known to the unknown…

…with some approximation in between.

A quote

When a problem is clearly stated, it is of no further interest to the physicist

P. DEBYE

What we need

- Some idea of how the world works.
  (We need to know something)
- General principles.
  (Conservation of energy, momentum, angular momentum,. . .)
  (Invariances,. . .)
- Simple systems.
  (Constant force, linear force — the spring,. . .)
  (Start simple, build up to complicated.)
- A system of units.
  (We need to compare measurements)

Another quote

Analysing a general potential system with two degrees of freedom is beyond the capabilities of modern science

V.I. ARNOL'D

Units

Observations — *measurements* — in physics always rely on comparison with some standards: we use this to define a set of physical units.

e.g. SI units: m, kg, s, mole, . . .

e.g. natural units: $\hbar = c = 1$.

e.g. Planck units: G, $\hbar$, c, $1/4\pi\epsilon_0$, k_B.

Aside: Planck Units

Tell us something fundamental?

Planck length	$l_P = \sqrt{\frac{\hbar G}{c^3}}$	1.616×10^{-35} m
Planck mass	$m_P = \sqrt{\frac{\hbar c}{G}}$	2.176×10^{-8} kg
Planck time	$t_P = \frac{l_P}{c} = \frac{\hbar}{m_P c^2} = \sqrt{\frac{\hbar G}{c^5}}$	5.391×10^{-44} s
Planck temperature	$T_P = \frac{m_P c^2}{k_B} = \sqrt{\frac{\hbar c^5}{G k_B^2}}$	1.416×10^{32} K

Scales

What sets the scale of a phenomenon?

e.g. size of atom, size of human.

Of course, we need some idea of different scales: facts.

(— why physics is not mathematics).

Calculating things

- Estimate the answer.
  (— order of magnitude calculations)
- Does it make sense? (— check)
- Improve if needed.

Problem solving: some ideas

Work from what we know.

- Scaling from a known state.

Model building.

- Dimensional analysis.

Visualisation.

- Simple computer simulations.

Intuition and heuristics.

- If all else fails, guess.

1. Using scaling

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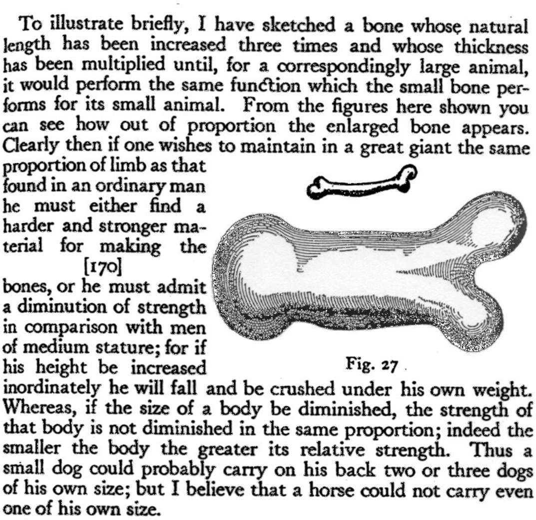
Use scale transformation to compare what we know with what we don’t.

Dates back to Galileo, who worried about the relative size of bones of animals.

Galileo (1638)



To illustrate briefly, I have sketched a bone whose natural length has been increased three times and whose thickness has been multiplied until, for a correspondingly large animal, it would perform the same function which the small bone performs for its small animal. From the figures here shown you can see how out of proportion the enlarged bone appears. Clearly then if one wishes to maintain in a great giant the same proportion of limb as that found in an ordinary man he must either find a harder and stronger material for making the bones, or he must admit a diminution of strength in comparison with men of medium stature; for if his height be increased inordinately he will fall and be crushed under his own weight. Whereas, if the size of a body be diminished, the strength of that body is not diminished in the same proportion; indeed the smaller the body the greater its relative strength. Thus a small dog could probably carry on his back two or three dogs of his own size; but I believe that a horse could not carry even one of his own size.

Fig. 27

[170]

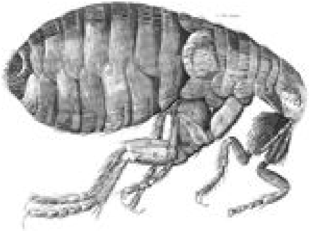
Question: How high can an animal jump?

How does the height to which an animal can jump depend on the size of an animal?

Big animal

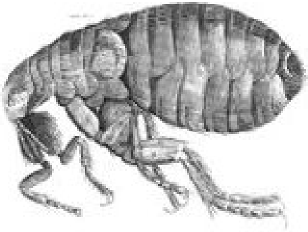
images: ocr.xepher.net

Small animal



images: Wikipedia

How high can they jump?



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Cautions

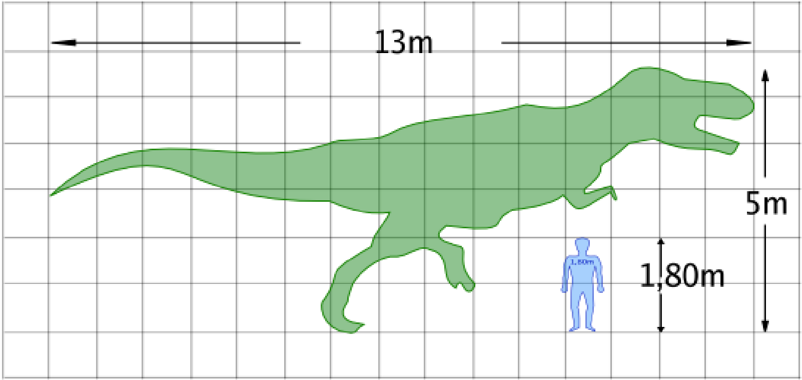
Of course, we have to be a bit careful about applying scaling in this situation.

We should only apply it to animals of the same basic structure.

Muscle structure and operation might vary considerably.

So think of Flea as a rather small lizard…

Animals



images: Wikipedia

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Rather than force, let's consider energy and work.

Energy in jumping

The energy comes from work done in the muscles.

We can write this as Fd where F is the force exerted by a muscle, and d is the distance over which the force acts.

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Then $Fd = c_1 AL$.

The height again

Using this we can determine the height h reached:

$$mgh = Fd = c_1 AL$$

and so

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Now, the mass is proportional to the density, which is also presumably a constant for all animals...

$$h = \frac{c_1}{g\rho} \frac{AL}{V}$$

The height again

Now we have a collection of constants, and some quantities which all depend on the size of the animal.

Suppose we scale up from Flea (L) to TRex ($L' = \alpha L$).

Then $A' = \alpha^2 A$, $V' = \alpha^3 V$, etc.

And the animal can jump . . .

Thus we find

$$h' = \frac{c_1}{g\rho}\,\frac{\alpha^2 A\alpha L}{\alpha^3 V} = \frac{c_1}{g\rho}\,\frac{AL}{V} = h$$

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Do you believe this?

Check!

OK, so some of you don’t believe this…

Check!

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Let's try a slightly different approach.

Energy in jumping

Let’s model the process as a spring. (Hooke’s law.)

The energy stored in a spring is

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We can use conservation of energy to find that the height reached is

$$mgh = \frac{1}{2} k \, (\delta x)^2$$

where m is the mass of the animal and $g = 9.8$m/s^2.

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F is force, A is cross sectional area, δL is extension, L is initial length. Y is a constant.

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F is force, A is cross sectional area, δL is extension, L is initial length. Y is a constant.

If all muscle is similar then Y the same for all animals, and everything else depends on the size of the animal!

The height again

Putting things in the form of Hooke's law we find

$$mgh = \frac{1}{2}\,\frac{AY}{L}\,(\delta L)^2 \quad \text{and so} \quad h = \frac{1}{2}\,\frac{1}{mg}\,\frac{AY}{L}\,(\delta L)^2$$

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$$h = \frac{Y}{2g}\,\frac{1}{\rho V}\,\frac{A}{L}\,(\delta L)^2$$

Scaling

Assuming all muscles work the same, the relative extension $\epsilon = \delta L/L$ must be the same for all animals.

Our expression for height is then

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Thus we find

$$h' = \frac{Y}{2g\rho}(\epsilon)^2 \frac{1}{\alpha^3 V} \alpha^2 A\alpha L$$

In other words,

$$h' = h$$

so that the height does not change at all!

Some data

	mass	h [m]	d_{launch} [m]	v_{launch} [cm/s]	a
flea	0.49 mg	0.20	0.00075	190	245 g
beetle	40 mg	0.30	0.0008	240	380 g
locust	3 g	0.6	0.04	340	15 g
man	70 kg	0.6	0.4	343	1.5 g

Some thoughts

How do we define jump height?

How do animal proportions scale?

What about other effects — e.g. air drag?

— refine the model.

Galileo (1638)

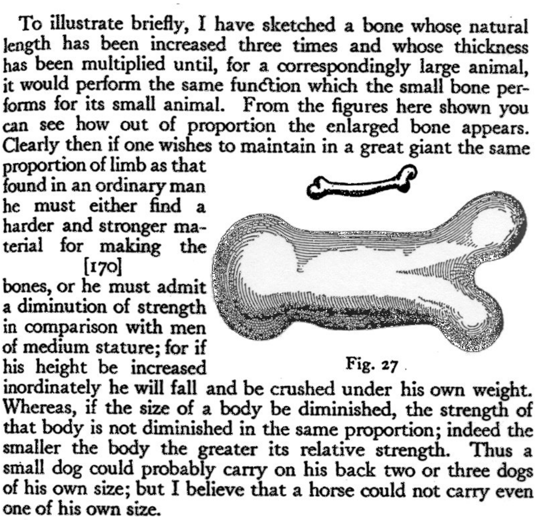

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Fig. 27

The story so far

Consider two animals (of same design?) of lengths L and L'.

Then the heights to which they can jump are related by

$$h' = h$$

so that the height does not change at all with size!

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That's wonderful! We love problems!

Air drag

What does air drag depend on?

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- Cross-sectional area. A.
- Density of air. ρ.
- Speed. v.
- Shape. A number. (Drag coefficient C_d).

Air drag

After some thought we can put these together to get a force

$$F_d = \frac{1}{2} C_d A \rho v^2$$

Since this increases with speed, it will eventually equal the gravitational force acting on a falling body.

Then the body will fall with constant speed.

$$mg = \frac{1}{2} C_d A \rho v_t^2 \quad \Longrightarrow \quad v_t = \sqrt{\frac{2mg}{CA\rho}}$$

More thoughts on air drag

Drag coefficient C_d. For sphere, often take $C_d = 0.5$.

But C_d depends on v, via Reynold's number.

$$C_d(v) \approx \frac{24}{R_n} + \frac{6.0}{1.0 + \sqrt{R_n}} + 0.4$$

where

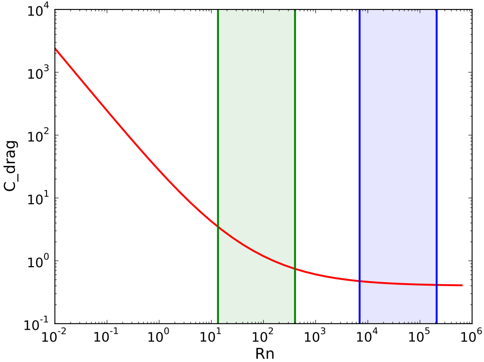
$$R_n = \frac{2r\rho v}{\eta}.$$

with ρ density of air (medium) and η is dynamical viscosity.

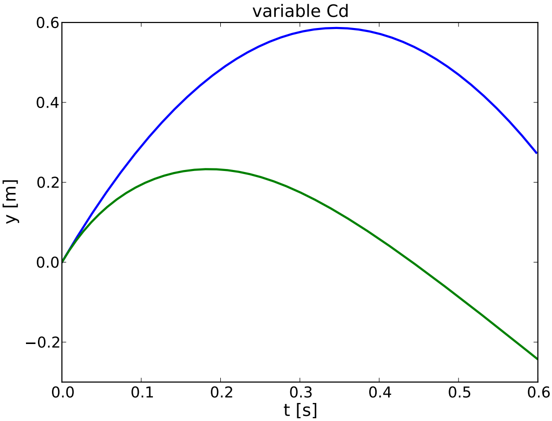
Air: $\rho = 1.17$ kg/m^3; $\eta = 1.85 \times 10^{-5}$ Pa s.

More thoughts on air drag

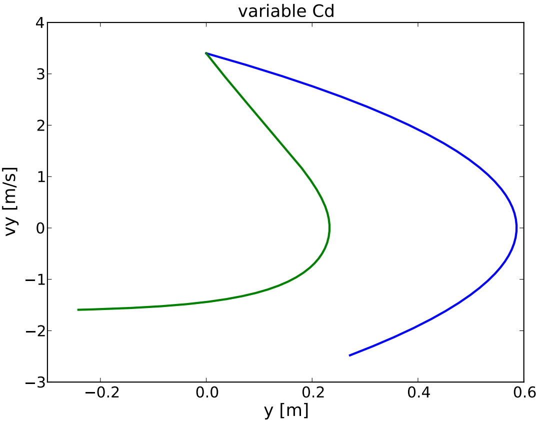
Reynold’s number is a dimensionless number, which characterises ratio of inertial forces (ρv) to viscous forces (η/r)



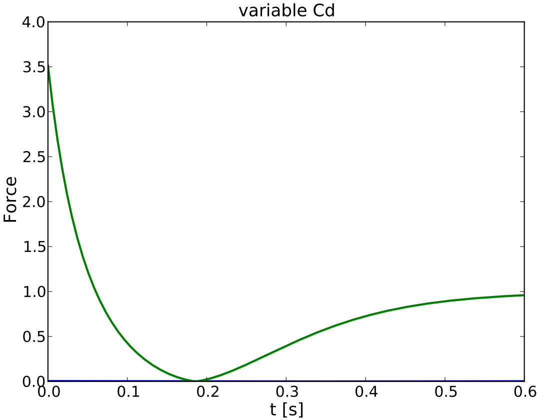
Man vs. flea: Jump



Man vs. flea: Speed



Man vs. flea: Drag force/grav force



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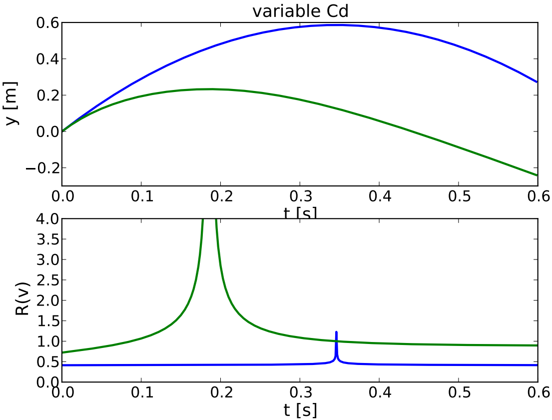

Rise or fall with airdrag

Newton's second law (N2):

$$\begin{aligned} \frac{dp_y}{dt} &= F_y \\ \frac{dy}{dt} &= \frac{p_y}{m} \end{aligned}$$

where $p_y = mv_y$.

We can solve Newton's 2nd law for a body falling with a constant drag coefficient:

$$m\frac{dv_y}{dt} = -mg - b|v_y|v_y = F_y(v)$$

But with a variable drag coefficient, we must use numerical integration.

Numerical integration

Write N2 as:

$$m\frac{dv_y}{dt} = -mg - b|v_y|v_y \quad \text{and} \quad \frac{dy}{dt} = v_y$$

Approximate derivatives:

$$\frac{dy}{dt} \approx \frac{y(t+\Delta t) - y(t)}{\Delta t} + \mathcal{O}(\Delta t)$$

Now N2 becomes:

$$v_y(t+\Delta t) = v_y(t) - [g - (b/m)|v_y(t)|v_y(t)]\,\Delta t \quad \text{and} \quad y(t+\Delta t) = y(t) + v_y(t)\Delta t$$

This tells us what happens next.

Numerical integration

Actually, should write this as

$$v_y(t+\Delta t) = v_y(t) - [g - (b/m)|v_y(t)|v_y(t)]\,\Delta t$$
$$y(t+\Delta t) = y(t) + v_y(t+\Delta t)\Delta t$$

We can try this in Python.

```
vy=vy-(g-(b/m)*fabs(vy)*vy)*dt
y =y +vy*dt
```

or

```
vy[i]=vy[i-1]-(g-(b/m)*fabs(vy[i-1])*vy[i-1])*dt
y[i] =y[i-1] +vy[i]*dt
```

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In 1 dimension:

$$M\frac{d^2x}{dt^2} = F(x)$$

Given the initial condition, the solution determines the trajectory of the system.

Scaling the coordinates

Consider the case where the force is a homogeneous function of order $k-1$:

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$$M\frac{\alpha}{\beta^2}\frac{d^2x'}{dt'^2} = \alpha^{k-1}F(x')$$

so the form is unchanged if $\beta = \alpha^{1-k/2}$

Scaling trajectories

The equations of motion are unchanged in form.

We can investigate how the trajectories change.

We find $\beta = \alpha^{1-k/2}$ so that:

Times scale as $\beta = \alpha^{1-k/2}$

Energies scale as α^k

Angular momentum scales as $\alpha^{1+k/2}$

etc.

Scaling example: energy

Many systems can be characterised by a conserved energy

$$E = K + U(x) = \frac{1}{2}\, m \left(\frac{dx}{dt}\right)^2 + U(x)$$

Suppose that the potential energy U satisfies

$$U(\alpha x) = \alpha^k\, U(x)$$

i.e. potential energy is homogeneous of degree k.

Examples

e.g. Harmonic oscillator

$$U(\alpha x) = \frac{1}{2}\,k_s\,(\alpha x)^2 = \alpha^2\,\frac{1}{2}\,k_s x^2$$

Gravitational potential

$$U(\alpha x) = \frac{k_g}{\alpha x} = \alpha^{-1}\,\frac{k_g}{x}$$

Scale time

We can also scale times in the problem, so that

$$t' \to \beta t \quad \text{when} \quad x' \to \alpha x$$

Then the velocity scales as

$$v' \to \left(\frac{\alpha}{\beta}\right) v$$

and kinetic energy scales as

$$K' = \frac{1}{2} m v'^2 \quad \to \quad \left(\frac{\alpha}{\beta}\right)^2 \frac{1}{2} m v^2$$

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$$\left(\frac{\alpha}{\beta}\right)^2 = \alpha^k$$

so that

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Scaling trajectories

Once again, we find $\beta = \alpha^{1-k/2}$ so that:

Times scale as $\beta = \alpha^{1-k/2}$

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etc.

Example

Take the harmonic oscillator , $U = (1/2)\, k_s x^2$, $k = 2$.

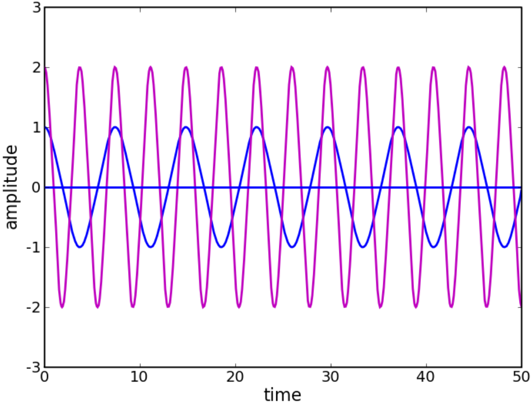
Energy scales as α^2: if we double the amplitude, the energy increases by 4.

Time scales as α^0: if we double the amplitude, the period remains unchanged!

So we can learn a lot about general properties of the system!

Another example: The quartic potential

$U(x) = x^4$; $k = 4$; t scales as $\beta = \alpha^{-1}$.



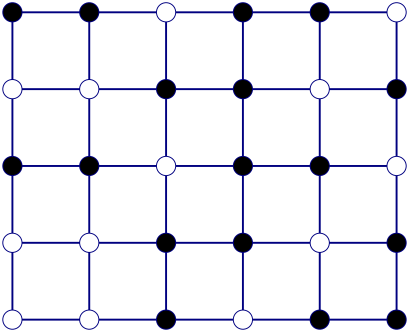
More scaling: Percolation

- Insulator-conductor transition on surface
- Percolation of water, oil through rocks
- Model for simple phase transitions

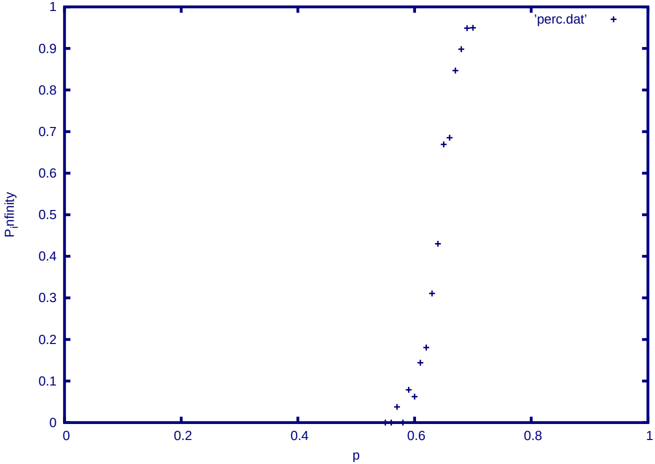
Site percolation

- Populate sites on lattice at random with probability p
- Forms clusters of sites
- Spanning cluster reaches all boundaries (spans lattice)
- Percolation threshold for formation of spanning clusters ($p > p_c$)
- At threshold, cluster size distribution is expected to be scale invariant, i.e. does not depend on scale.
  We can use this to calculate the threshold for some lattices.

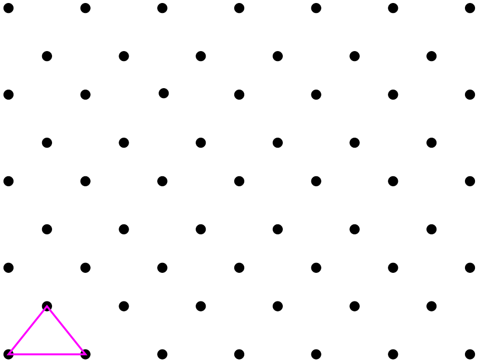
Percolation on a (square) lattice



Percolation threshold



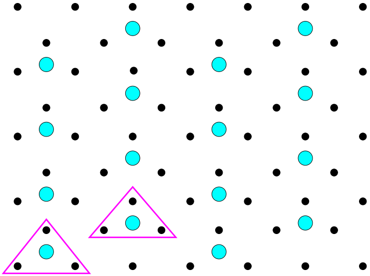
Percolation on a (triangular) lattice



Percolation on a (triangular) lattice

Scale invariance at percolation threshold means probability of occupying a site $p = p_c$ is independent of lattice size.

Rescale lattice to group three points into one.



Percolation on a (triangular) lattice

Scale invariance at percolation threshold means probability of occupying a site $p = p_c$ is independent of lattice size.

Rescale lattice to group three points into one.

Probability of occupying a site is p'.

$$p' = p^3 + 3\,p^2\,(1 - p)$$

But $p = p' = p_c$.

thus $p_c = 1/2$ (other roots at $0, 1$ are trivial).

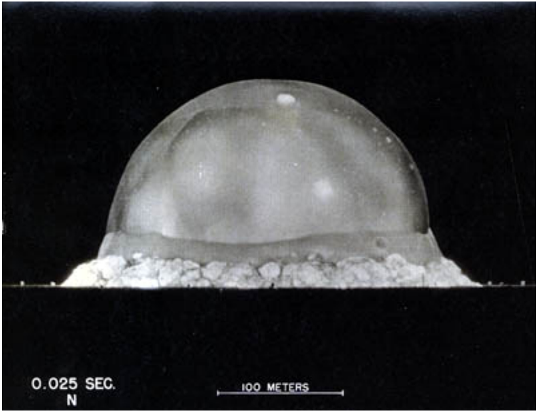
2. Dimensional analysis

We can often figure out a solution if we know what physical variables are involved.

These variables are all associated with fundamental quantities like length, time, mass, etc. These are known as the “dimensions” of a quantity, and are usually standardised though a system of units.

We can analyse these “dimensions” to determine the form of the solution.

An example!



16 July 1945. G. I Taylor estimated blast energy.

A problem!

A (marked!) N_2 molecule is released in the centre of the room.

How long does it take to get to a wall?

A problem!

A (marked!) N_2 molecule is released in the centre of the room.

How long does it take to get to a wall?

Guess?

A problem!

A (marked!) N_2 molecule is released in the centre of the room.

How long does it take to get to a wall?

Guess?

How do we go about solving this?

Questions

Is there anything special about an N_2 molecule?

Questions

Is there anything special about an N_2 molecule?

How far does it go?

Questions

Is there anything special about an N_2 molecule?

How far does it go?

How fast does it move?

Questions

Is there anything special about an N_2 molecule?

How far does it go?

How fast does it move?

How does it move?

The room

How far does it go?

The speed

How fast does it move?

The speed

How fast does it move?

What can the speed depend on?

The speed

How fast does it move?

What can the speed depend on?

Equipartition of energy tells us something about the kinetic energy of the molecule.

$$K = \frac{1}{2} m v^2 = \frac{3}{2} kT$$

where $k = 1.4 \times 10^{-23}$ J/K is Boltzmann's constant.

The speed

How fast does it move?

What can the speed depend on?

Equipartition of energy tells us something about the kinetic energy of the molecule.

$$K = \frac{1}{2} m v^2 = \frac{3}{2} kT$$

where $k = 1.4 \times 10^{-23}$ J/K is Boltzmann's constant.

(appeal to some general principle of physics)

The motion

So how does the air molecule move through the air?

The motion

So how does the air molecule move through the air?

diffusion

So how do we solve a problem when we know so little?

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We will look at Dimensional analysis.

We can often argue from the dimensions of the quantity we want, and the quantities it depends on, to estimate a value.

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We can often argue from the dimensions of the quantity we want, and the quantities it depends on, to estimate a value.

We need to know something about units, which give values to these dimensions.

Units

Physics is based on measurement and experiment.
Units are required in order to express these measurements in a standard way.

Base units for basic quantities like distance, time and mass.

Derived units convenient combinations of base units.

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Base units for basic quantities like distance, time and mass.

Derived units convenient combinations of base units.

We use the S.I system of units.

Base units

Length

Sizes, lengths, distances are measured in metres (abbreviated m). (L)

Time

Time intervals and durations are measured in seconds (abbreviated s). (T)

Mass

Masses are measured in kilograms (abbreviated kg). (M)

Base units

Length

Sizes, lengths, distances are measured in metres (abbreviated m). (L)

Time

Time intervals and durations are measured in seconds (abbreviated s). (T)

Mass

Masses are measured in kilograms (abbreviated kg). (M)

Other base units: ampere, kelvin, mole, candela.

Base units

Temperature

Measure T in kelvin (abbreviated K) 0 K=-273.17° C.

Current

Electrical current is measured in amperes. From this we derive the unit for charge, measured in coulombs (abbreviated C).

Mole

Measures amount of substance
(abbreviated mol). 1 mol $= 6 \times 10^{23}$ things

Base units: Length, Time, Mass

The metre is defined as the distance light travels in vacuum in $1/299792458$ seconds.

Note that the speed of light is now,
by definition, $c = 299792458$ m/s.

The second is defined using the characteristic transition frequency of a particular kind of caesium atom in an atomic clock, as 9192631770 periods.

The kilogram is defined as the mass of a certain platinum-iridium alloy cylinder kept at the International Bureau of Weights and Measures in France.

Derived units

Derived units are convenient combinations of the base units.
Often they are given their own names.

The combinations of units are termed the dimensions of the unit: the base units have dimensions of L, T and M.

The phrase [energy] can be read as “the dimensions of energy”.

Derived units: example

For example, we have

$$[\text{energy}] = \frac{[\text{mass}][\text{length}]^2}{[\text{time}]^2} = \frac{\text{ML}^2}{\text{T}^2}$$

The unit of energy is the joule, abbreviated J.

Note that we can treat the combination of units algebraically.

Use of units

All quantities in physics have units associated with them.
In some, the combinations of base units all cancel out; such quantities are termed dimensionless.

All quantities must thus be given with their units; they are otherwise meaningless numbers.

For example, a distance might be expressed as 10.42 m.

Use of units (2)

Two quantities can only be equated if they have the same units.

We can use this as a rough check on the consistency of our equations:

$$x = \frac{1}{2}\,a\,t^2$$

$$[x] = [a][t^2]$$

$$\mathrm{L} = \frac{\mathrm{L}}{\mathrm{T}^2}\,\mathrm{T}^2$$

Units: important points

- A number referring to a physical quantity is meaningless without units.
- Both sides of an equation must have the same units.
- Use powers of 10 and prefixes to scale units.

Units: SI prefixes

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{24}	yotta	Y	10^{-1}	deci	d
10^{21}	zetta	Z	10^{-2}	centi	c
10^{18}	exa	E	10^{-3}	milli	m
10^{15}	peta	P	10^{-6}	micro	μ
10^{12}	tera	T	10^{-9}	nano	n
10^{9}	giga	G	10^{-12}	pico	p
10^{6}	mega	M	10^{-15}	femto	f
10^{3}	kilo	k	10^{-18}	atto	a
10^{2}	hecto	h	10^{-21}	zepto	z
10^{1}	deka	da	10^{-24}	yocto	y

So, back to our problem!

A (marked!) N_2 molecule is released in the centre of the room.

How long does it take to get to a wall?

We will solve this using dimensional analysis

Some hints on solving problems

- Use whatever information you have.
- Use general principles to guess what you don’t know.
- Make a plausible guess if there is no general principle.
- Decide which variables are relevant.
- Exploit units.
- Check that the expression that you end up with behaves sensibly.
- Do the numbers seem right?

Some general principles

- Conservation of energy and momentum
- Other conservation laws
- Symmetries
- Statistical or thermal equilibrium
- Equipartition of energy
- Central limit theorem

Some facts

We need to know some basic facts about the world.

Size of an atom?

Number of atoms per cubic metre?

Density of air?

How far. . .

How far does our N_2 molecule move between collisions?

This is called the mean free path.

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What can it depend on?

Mean free path

Mean free path

Size of molecule?

Mean free path

Size of molecule?

Number of other molecules around?

Mean free path

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How fast it/they move?

Mean free path

Size of molecule?

Number of other molecules around?

How fast it/they move?

Mass of the molecules?

Mean free path

Size of molecule?

Number of other molecules around?

How fast it/they move?

Mass of the molecules?

Temperature?

Mean free path

Size of molecule?

Number of other molecules around?

How fast it/they move?

Mass of the molecules?

Temperature?

Time of day?

Mean free path

Suppose that mean free path λ depends on

Size of molecule d

Number of other molecules around N

Mean free path

Suppose that mean free path λ depends on

Size of molecule d

Number of other molecules around N

How does mean free path depend on these quantities?

Mean free path

We have $[d] = L$, $[N] = L^{-3}$.

The simplest way we can get a quantity of dimension L is

$$\lambda = \frac{1}{Nd^2}$$

Mean free path

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The simplest way we can get a quantity of dimension L is

$$\lambda = \frac{1}{Nd^2}$$

Does this make sense?
(Make a sanity check on your results)

Mean free path

We can imagine a molecule of cross sectional area $A = d^2$ moving through a random distribution of points.

It moves a distance λ and sweeps out a volume λd^2.

The number of scatterers in the volume is $N\lambda d^2$.

On the average you would expect the number to be 1.

Hence, $\lambda = \frac{1}{Nd^2}$.

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On the average you would expect the number to be 1.

Hence, $\lambda = \frac{1}{Nd^2}$.

(What about the size of the scatterers?)
(What about the speed of the scatterers?)

Mean free path

We estimate:

Size of molecule is $\sim 10^{-10}$ m.

Let's take $d^2 = 10 \times 10^{-20}$ m^2.

We know that the molar mass of nitrogen is 28 g.
Hence the number of molecules per unit volume is approximately

$$N = \frac{\rho}{28} N_A \approx 40 N_A \approx 2.4 \times 10^{25} \quad \text{per m}^3$$

So,

$$\lambda = \frac{1}{N d^2} \approx 4 \times 10^{-7} \text{ m}$$

Back to the problem!

So, we can understand that the distance a molecule moves between collisions is very small.

What can the time T depend on?

Back to the problem!

So, we can understand that the distance a molecule moves between collisions is very small.

What can the time T depend on?

L, N, d, v.

Time only appears in v

Possible forms of the solution

We can play around with the variables we have and try to come up with a form having the right units.

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$$T = \frac{NL^2d^2}{v}$$

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$$T = \frac{L}{v}$$

$$T = \frac{NL^2d^2}{v}$$

$$T = \frac{NdL^3}{v}$$

Solving the problem

After some sanity checking, we settle on

$$T = \frac{Nd^2L^2}{v} = \frac{L^2}{v\lambda}$$

Putting in some numbers ($L = 5$ m, $v = 500$ m/s) we find

$$T \approx 2.5 \times 10^5 \text{ s}$$

or 3 days.

Is this reasonable?

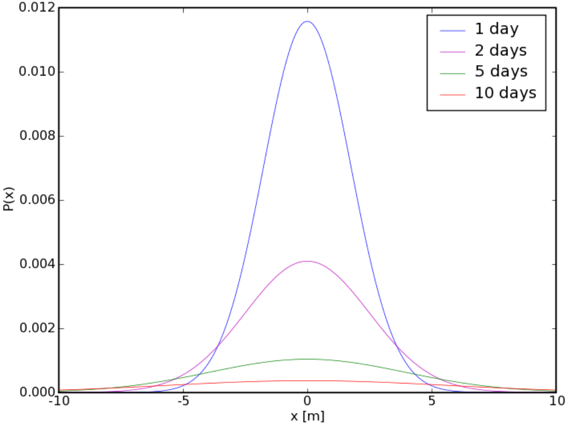
Diffusion constant $D = v\lambda/3$.

Diffusion:

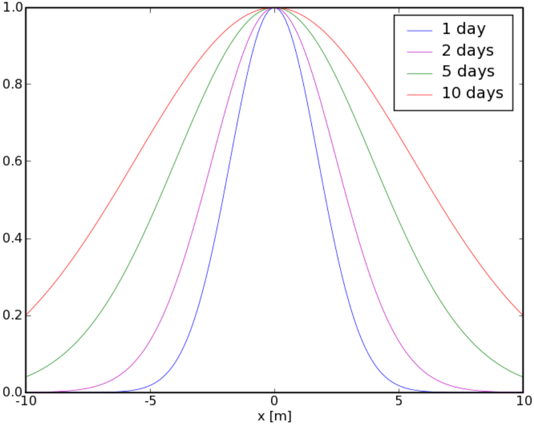
$$P(r)dr = \frac{1}{(4\pi Dt)^{3/2}} e^{-\frac{r^2}{4Dt}} dr$$

For self diffusion of nitrogen, $D = 0.18 \times 10^{-4}$ m^2/s;
our calculation is $v\lambda = 2 \times 10^{-4}$ m^2/s

Diffusion


0.012
0.010
0.008
0.006
0.004
0.002
0.000
P(x)
-10
-5
0
5
10
x [m]
1 day
2 days
5 days
10 days

Diffusion



More comments

Kinetic theory informs us that the distribution of molecular speeds in the gas is given by:

$$F(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \; v^2 \, e^{-\frac{mv^2}{2kT}}$$

More comments

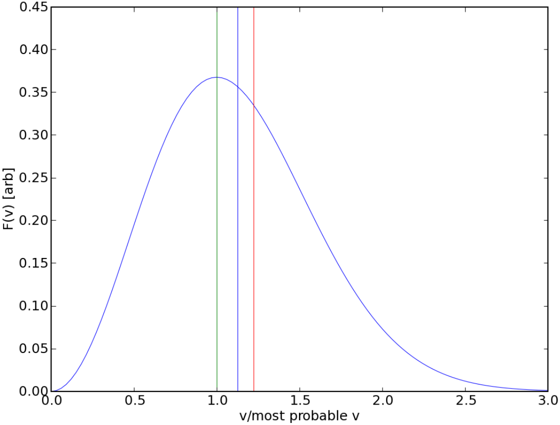
We can define 3 “average” speeds:

$$\bar{v} = \sqrt{\frac{8}{\pi}\frac{kT}{m}}$$ mean speed

$$\tilde{v} = \sqrt{2\frac{kT}{m}}$$ most probable speed

$$v_{\mathrm{rms}} = \sqrt{3\frac{kT}{m}}$$ root mean square speed

Maxwell speed distribution



Dimensional analysis

We look at some of the formal theory of this subject.

We want to associate some quantity, say force F, with a set of variables which we decide are relevant, e.g. mass m and acceleration a.

We want to relate these: we write a general relationship in the form

$$f(F, m, a) = 0$$

Dimensionless groups

Our quantities have units:

$$
\begin{aligned}
[F] &= MLT^{-2} \\
[m] &= M \\
[a] &= LT^{-2}
\end{aligned}
$$

From these we can construct dimensionless groups, e.g

$$\frac{F}{ma}, \quad \frac{ma}{F}, \quad \pi \left(\frac{F}{ma}\right)^{42}, \quad \ldots$$

that have no units.

Dimensionless groups

The quantities we seek are solutions (roots) of the equation

$$\hat{f}\left(\frac{F}{ma}\right) = 0$$

If we have chosen our variables well, then the root will have a value near 1, e.g

$$\frac{F}{ma} = 1, \quad \frac{F}{ma} = \frac{\pi}{2}, \quad \ldots$$

If we cannot easily find the correct value, we choose 1.

Buckingham Pi theorem

Suppose we have an equation in N variables:

$$f(q_1, q_2, q_3, \ldots, q_N) = 0$$

and the q_i are expressed in terms of m independent quantities.
Then we can describe the system by $k = N - m$ dimensionless quantities

$$\hat{f}(\Pi_1, \Pi_2, \ldots, \Pi_k) = 0$$

e.g.
3 variables F, m, a
2 dimensions M and LT^{-2}
1 dimensionless group F/ma.

Simulation

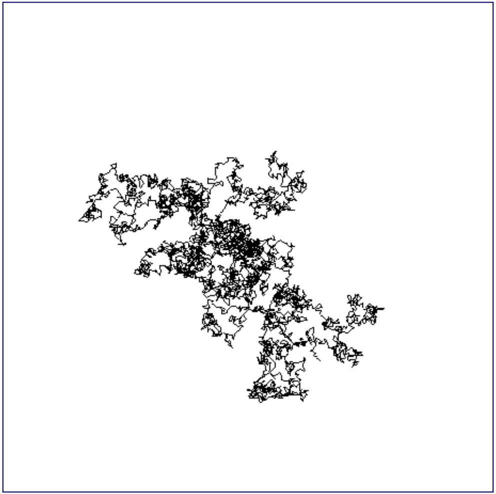
Simulating the diffusion process

A random stroll through random walks…

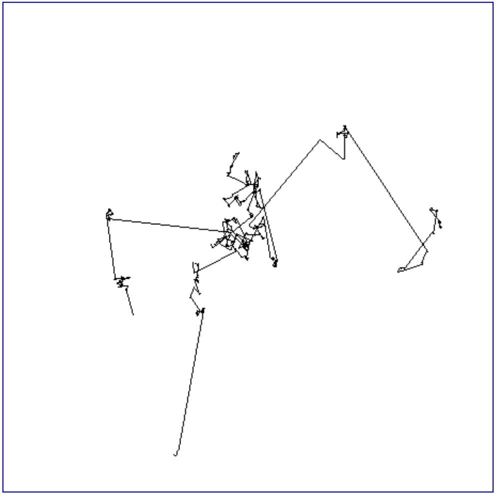
A small particle buffeted by atoms in a gas can be observed (Brown, 1828) to move along a randomly changing path. This Brownian motion is an example of a random walk.

A drunkard walks out of the front door of AIMS and staggers randomly left or right.
How far does he get after N steps?

Random walk: example 1



Random walk: example 2



One dimension

Suppose a drunkard takes steps s_i with equal probability $1/2$ to the left and right, of size $\pm d$. The position after N steps is

$$x(N) = \sum_{1}^{N} s_i$$

The mean $< x(N) >= 0$.
The displacement squared is

$$x^2(N) = \left(\sum_{1}^{N} s_i \right)^2 = \sum_{1}^{N} s_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} s_i s_j$$

One dimension (2)

Since s_i is $\pm d$ with equal probability, the second term averages to zero; then

$$x^2(N) = \sum_1^N d^2 = Nd^2$$

Thus the variance

$$\langle (\Delta x)^2(N) \rangle = \langle x^2(N) \rangle - \langle x(N) \rangle^2$$

is proportional to N.

Binomial distribution

Walker starts from the origin, moves right with probability p and left with probability $q = 1 - p$.

After N steps, n steps to the right and $m = N - n$ steps to the left. Also, sum over the number of different walks.

The probability density for being n steps to the right after a total of N steps is

$$P(N, n) = \binom{N}{n} p^n \, q^{N-n} = \frac{N!}{n!(N-n)!} \, p^n \, q^{N-n}$$

— the binomial distribution.

Recall the binomial expansion:

$$(p + q)^N = \sum_{n=0}^{N} \binom{N}{n} p^n \, q^{N-n}$$

Master equation

Situation after $N+1$ steps.
Walkers previously at n have left (with probability $p+q(=1)$ while walkers at $n-1$ or $n+1$ have moved to n with probability p or q respectively.
Hence

$$P(N+1,n) = P(N,n) - (p+q)P(N,n) + pP(N,n-1) + qP(N,n+1)$$

Diffusion

Replace the variables $N \to t = N\delta t$ and $n \to x = n\delta x$.
Suppose N and n are large so that we may treat the problem as continuous.
Then

$$P(t + \delta t, x) = pP(t, x - \delta x) + qP(t, x + \delta x)$$

Expanding in Taylor series, and choosing $p = q = 1/2$

$$\frac{\partial P(t,x)}{\partial t} = \frac{(\delta x)^2}{2\delta t}\frac{\partial^2 P(t,x)}{\partial x} \equiv D\frac{\partial^2 P(t,x)}{\partial x}$$

Diffusion equation, with diffusion coefficient D.

Diffusion — solution

The solution to diffusion equation, with initial condition $P(0, x) = \delta(x)$ is

$$P(t, x) = \frac{1}{\sqrt{4\pi Dt}} \exp - \frac{x^2}{4Dt}$$

Show that the limit for large N of the binomial distribution is also the Gaussian.

Continuous random walk

Suppose that the displacement of a step is a random variate taken from a distribution.

The probability of finding step size s is $p(s)ds$.

Thus the probability of finding the walker in $(x, x + dx)$ after one step is

$$P(1, x)\, dx = p(x)\, dx$$

After two steps this becomes

$$P(2, x)\, dx = p(x_1)\, dx_1\, p(x_2)\, dx_2$$

subject to the constraint $x_1 + x_2 = x$.

Continuous random walk —(2)

Handle constraint with a delta function:

$$\begin{aligned} P(2,x)\,dx &= \int p(x_1)\,dx_1\, p(x_2)\,dx_2\,\delta(x-x_1-x_2) \\ &= \int p(x-x2)\,p(x_2)\,dx_2 \end{aligned}$$

The probability density is the convolution of the two single step densities.
This may be extended to any number of steps.
Provided that $p(x)$ has a finite second moment, the central limit theorem tells us that $P(N,x)$ tends to Gaussian distribution after many steps N.

Random numbers

Simulate physical situation using (pseudo)random numbers.

Average to obtain results.

Must be able to generate “random” numbers on the computer.
These numbers will be computed by some deterministic process, but the sequence will have such properties that we can regard it as random.

Generating a random number

is tricky…

e.g. the linear congruential generator.
Generates a sequence of integers $N_1, N_2, \ldots$ between 0 and m with

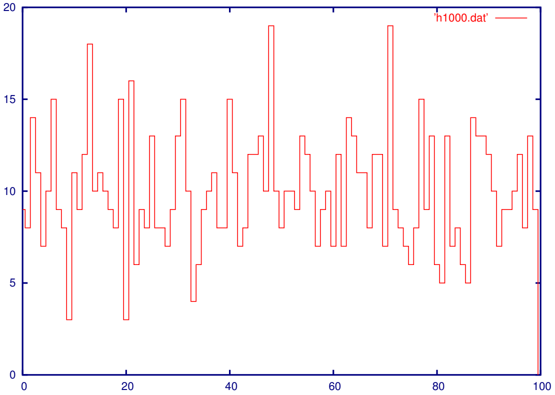
$$N_{i+1} = (aN_i + c) \mod m$$

where a, c and m are constants. N_0 is the seed.
e.g. $a = 16807$, $c = 0$, $m = 2^{31} - 1$.
(infamous: $a = 65539$, $c = 0$, $m = 2^{31}$)$_{\text{randu}}$

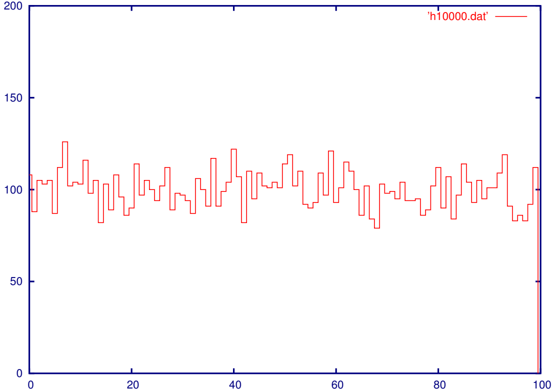
A random number in $(0, 1)$ can be obtained by dividing by m.

Uniform random numbers

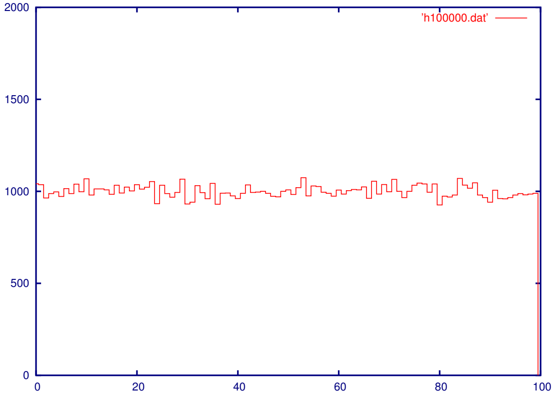
Uniformly distributed in $(0, 1)$.



Uniform random numbers



Uniform random numbers



Non-uniform random numbers

It is often required to have random numbers from some other distribution than uniform (on $(0, 1)$).

These can often be generated with the aid of uniform random numbers.

Exponential distribution

$$p(y) = \begin{cases} \frac{1}{\lambda} e^{-y/\lambda} & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

Typical of Poisson processes.

e.g. path lengths between collisions in a gas, or waiting times in radioactive decay.

Exponential distribution

Then

$$x = P(y) = \int_0^y \frac{1}{\lambda} e^{-y'/\lambda}\, dy' = 1 - e^{-y/\lambda}$$

and

$$y = -\lambda \ln(1 - x)$$

where x is drawn from a uniform distribution.

Normal distribution

A common technique to generate normally distributed random numbers is the Box-Muller transform.

A generalisation of the transformation technique to two dimensions can be applied to yield two normal deviates from two uniform deviates.

$$\begin{aligned} y_1 &= \sqrt{-2\ln x_1}\,\cos 2\pi x_2 \\ y_2 &= \sqrt{-2\ln x_1}\,\sin 2\pi x_2 \end{aligned}$$

are independent Gaussian random variables with zero mean and unit varia

Simulation

A simple algorithm in 2d is

```
Start at the origin.
For N steps:
    Compute a random x and y displacement
    Use this to calculate the next position
Repeat M times and compute a histogram of the endpoints.
From this, compute whatever moments of the
distribution are required.
```

Random numbers in Python

Random numbers in $(0, 1)$.

```
from visual import *
from numpy.random import random

x=random()
y=random(N)
```

Random decisions in Python

```
from visual import *
from numpy.random import random

x=random()
if x<p:
    ... do something with probability p ...
```