Exercises 5

1. Let $I = \langle g_1, g_2, g_3 \rangle \subset \mathbb{R}[X, Y, Z]$, where $g_1 = XY^2 - XY + Y$, $g_2 = XY - Z^2$ and $g_3 = X - YZ^4$. Using the lex order (with $X > Y > Z$), give an example of $g \in I$ such that $\text{LT}(g) \notin \langle \text{LT}(g_1), \text{LT}(g_2), \text{LT}(g_3) \rangle$.

2. Is $G = \{X^4Y^2 - Z^5, X^3Y^3 - 1, X^2Y^4 - 2Z\}$ a Gröbner basis for the ideal $\langle G \rangle$ with respect to the grlex order?

3. Let $I \subset K[X_1, X_2, \cdots, X_n]$ be a principal ideal. Show that any finite subset of I containing a generator is a Gröbner basis for I.

4. Let $f \in K[X_1, X_2, \cdots, X_n]$. If $f \notin \langle X_1, X_2, \cdots, X_n \rangle$ then show that

$$\langle X_1, X_2, \cdots, X_n, f \rangle = K[X_1, X_2, \cdots, X_n].$$

5. Consider the variety $V = \mathbf{V}(X^2 - Y, Y + X^2 - 4) \subset \mathbb{C}^2$. Note that $V = \mathbf{V}(I)$, where $I = \langle X^2 - Y, Y + X^2 - 4 \rangle$.
   (a) Prove that $I = \langle X^2 - Y, X^2 - 2 \rangle$.
   (b) Using the basis from part a, find $\mathbf{V}(I)$.

6. Compute the S-polynomial of $f = x^2y + z$, $g = xz + y$ in $\mathbb{Q}[x, y, z]$ with respect to lex with $x > y > z$.

7. By making use of S-polynomials, show that the set $G = \{g_1, g_2, g_3\}$ from Question 1 is not a Gröbner basis for the ideal I.

8. We have seen that $G = \{x + z, y - z\}$ is Gröbner basis for lex order $x > y > z$.
   (a) Divide xy by $x + z, y - z$.
   (b) Divide xy by $y - z, x + z$.

   What do you notice?

9. Does $S(f, g)$ depend on which monomial order is used. Illustrate your assertion with examples.

10. Is the set $G = \{x^2 - y, x^3 - z\}$ a Gröbner basis for the ideal it generates using $grlex$.

11. Determine a Gröbner basis for $I = \langle -x^3 + y, x^2y - z \rangle$.

    (a) Using *grlex* with $y > x > z$.

    (b) Using *lex* with $x > y > z$.

12. Determine whether $f = xy^3 - z^2 + y^5 - z^3 \in I$ in Question 10.

13. By computing a Gröbner basis with respect to *lex* order with $x > y > z$ and solving the resulting polynomial equations obtained, find the solutions in $\mathbb{R}^3$ to the system of equations:

    (a)

$$\begin{aligned} x^2y - z^3 &= 0 \\ 2xy - 4z &= 0 \\ z - y^2 &= 0 \\ x^3 - 4yz &= 0 \end{aligned}$$

    (b)

$$\begin{aligned} x^2 + y^2 + z^2 - 1 &= 0 \\ x^2 + y^2 + z^2 - 2x &= 0 \\ 2x - 3y - z &= 0 \end{aligned}$$

14. Find a Gröbner basis for $I = \langle x^2y + z, xz + y \rangle \subset \mathbb{Q}[x, y, x]$ with respect to *lex* order with $x > y > z$. Decide whether each of the following belong to I.

    (a) $f = y^4 + y^2z^3 + 2z^2 - 2xy^2$

    (b) $g = x^3z + 3x^2y + 2z$

    Do the same with *grlex* order with $x > y > z$ (or $y > x > z$).

15. Determine whether $f = x^3z - 2y^2$ is the ideal $I = \langle xz - y, xy + 2z^2, y - z \rangle$. What happens if f is replaced by $f - 2z$?