Random numbers in Financial Mathematics:
Valuing Financial Options

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# 1 Generating (Pseudo-)Random Numbers

- Many statistical packages have (normal) random number generators
- Otherwise, generate Normal random numbers by:
  - generating random numbers that are uniformly distributed on $[0, 1]$.
  - Transforming them to obtain normally distributed random numbers
  - The other issue is that these are only "pseudo" random numbers in that the computer program typically has an algorithm for calculating the 'random numbers' , as John von Neumann said in 1951 "*Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin*".
  - There are many alternatives (The "Recipe" books of Press, Teuklovsky, Vetterling and Flannery are a good source, and have a good discussion of the problem).

## 1.1 The Box-Muller method

- The simplest technique for generating 'decent' Normally distributed random numbers is the Box-Muller method which takes any uniformly distributed variables (from any source you prefer) and turns them into Normally distributed ones.
- Given two uniformly distributed random numbers $x_1$, $x_2$, two Normally distributed random numbers, $y_1$ and $y_2$ are given by:

$$y_1 = \cos(2\pi x_2)\sqrt{-2\log(x_1)}, \quad y_2 = \sin(2\pi x_1)\sqrt{-2\log(x_2)}$$

# 2 Brownian motion and the model of stock price movement

This discrete random walk is one possible way of modelling stock price movements. It is, however, very simplistic and only works for discrete time.

The latter of these problems can be easily overcome by using its continuous time analogue: Brownian motion, or as it's sometimes referred to, the Wiener process.

The stochastic model of Brownian motion was, obviously, defined to mirror the movement of tiny particles in water but has applications in more fields than that, one being in option pricing theory. There is a lot of rigorous mathematics surrounding such processes but as regards this course a heuristic overview will be provided.

**Definition (Brownian motion)** A real valued stochastic process $W_t$ is a Brownian motion (or Wiener process) under a probability measure P if

1. For each $t \geq 0$ and $s > 0$ the random variable $W_{t+s} - W_t$ (often termed $dW$) is distributed Normally with mean 0 and variance $s$.

2. For each $n$ and for any times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ the random variables $\{W_{t_r} - W_{t_{r-1}}\}$ are independent.

3. $W_0 = 0$ (this is merely a convention, it can start from any point).

4. $W_t$ is continuous in $t \geq 0$.

This is basically just an extension of the discrete simple random walk to continuous time. The change $W_{t+s} - W_t$ over a very small period of time $dt$ is often denoted by $dW$ and obviously is distributed accordingly (mean of zero and variance of $dt$). Brownian motions obviously have very strange paths and, in fact, the expected length of path followed by $W$ in a any time interval is infinite, this will make calculus difficult on Brownian motions.

One way of understanding $dW$ is to see it as $\epsilon\sqrt{dt}$ where $\epsilon$ is distributed normally with a mean of 0 and variance of 1. The standard Brownian motion $W_t$ will not model stock prices very well for two main reasons:

- The general trend of stock prices is upwards whereas the expected movement of Brownian motion is to stay at the same level.

- Stock prices cannot drop below 0 whereas Brownian motion can take any real value.

### 2.0.1 Generalised Brownian motion

The first of these concerns can be overcome easily. On top of the random increments generating by the Brownian motion term it is possible to add in deterministic terms. When dealing with stock prices there is a general upward drift, call this $\mu$ and so if the stock price is denoted by $S_t$ then we have the following **stochastic differential equation**

$$S_{t+dt} - S_t = dS = \mu dt + \sigma dW \tag{1}$$

where $\sigma$ is just scaling the effect of Brownian motion. So, in this case over a period of time $dt$ the stock price increases from $S_t$ by an amount $\mu dt$ plus an unknown amount $\sigma dW$ where $dW$ is a Brownian motion. The distribution of the stock price increases, $dS$, is slightly different in this case

$$\begin{aligned} E[dS] &= E[\mu dt + \sigma dW] \\ &= \mu dt + \sigma E[dW] \\ &= \mu dt \end{aligned}$$

and similarly the variance is $\sigma^2 dt$ as before. This is an improvement on Brownian motion but still has the problem that there is nothing to prevent $S_t$ from dropping below zero.

### 2.0.2 Geometric Brownian motion

To over come this adapt the above process ever so slightly to

$$dS = \mu S dt + \sigma S dW. \tag{2}$$

This is saying that both the deterministic and random terms are scaled depending on the size of $S$ at time $t$. The larger $S$ is the bigger, on average, its movements are. This makes sense as a share with price \$3 is more likely to move by a cent than one worth 2 cents. More importantly, $S \geq 0$; as soon as $S = 0$ then the process remains there as $dS \equiv 0$

This process does not give rise to increments which are distributed Normally but rather ones which are distributed **lognormally**.

# 3 What are options?

**Definition (Call options)** A call option gives the holder the right, but not the obligation to buy the underlying, $S$, at a certain date, $T$, for a certain price, known as the exercise (or strike) price, $X$.

**Definition (Put options)** A put option gives the holder the right, but not the obligation to sell the underlying, $S$, at a certain date, $T$, for a certain price, known as the exercise (or strike) price, $X$.

There are also two main genres of options:

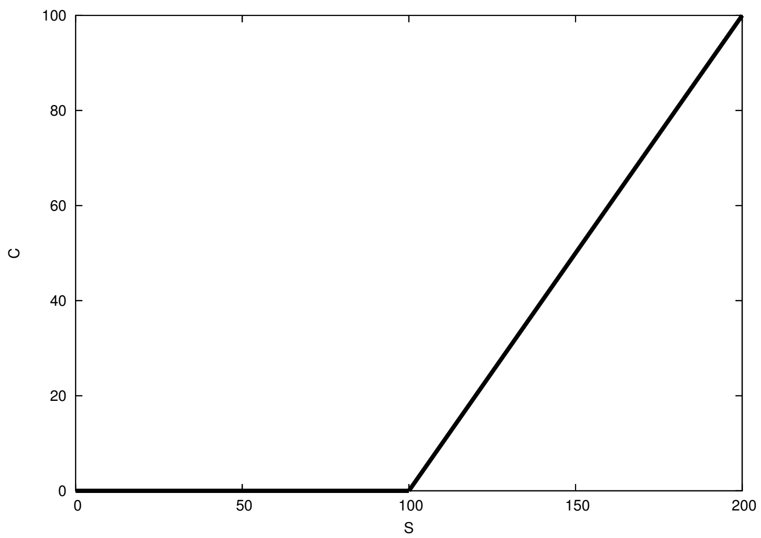
**Definition (European options)** A European option can can only be exercised at the expiration date $T$.

**Definition (American options)** An American option can be exercised itat any time up to and including the expiry date, $T$. These are much more difficult to value than European options!

Clearly the two parties (the holder and writer) have unequal 'rights' - therefore there is value in the contract, for which the writer (who must honour his/her obligation) must be paid. It is setting a fair value for this that option pricing is all about.

## 3.1 European Call Options

Denote the value of a European call option by $C(S, t)$ where $S$ is the value of the underlying asset at time $t$. If the strike price of the option is $X$ then at the expiry of the option, $t = T$, the holder of the option has the




right, but not the obligation, to buy the underlying, of value $S$ at $t = T$ at this price, $X$. Clearly if $S > X$ then the holder of the option would exercise the option and buy the underlying (worth $S$) for $X$. This would yield the holder of the option a profit of $S - X$. If $S \leq X$ then there is no point in exercising the option as the holder can buy the underlying on the market for less than $X$. Hence at expiry ($t = T$) the value of *the call option* is

$$C(S,T) = \max(S - X, 0).$$

### 3.2 European Put Options

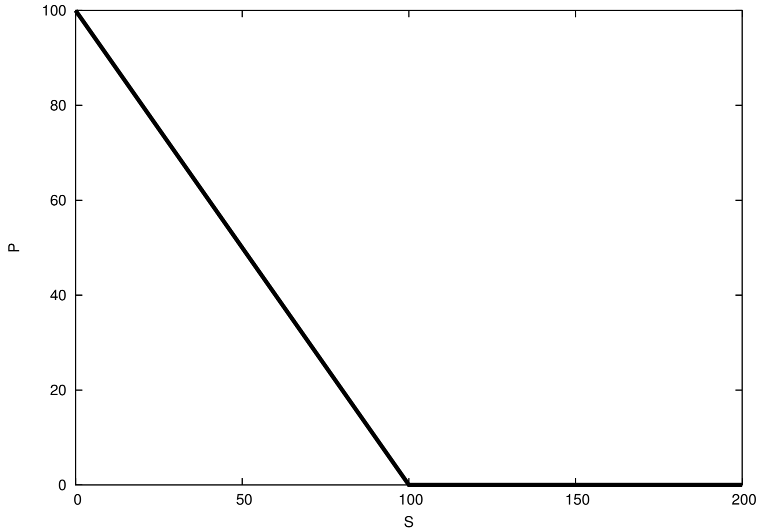
In a similar way to call options, denote the value of a put option by $P(S,t)$.Again the option has a strike price of $X$ and at expiry the holder of the option has the right, but not the obligation, to *sell* the underlying asset at this price. With a put option at expiry ($t = T$) $S < X$ then the holder of the option would exercise as she can sell the underlying for more than she could on the market, and the option would then be worth $X - S$. If, however, $S > X$ the the holder of the options could sell the underlying for more than $X$ and thus it would not be worth exercising the option.

Hence, at $t = T$ the value of a put option is

$$P(S,T) = \max(X - S, 0)$$

### 3.3 Why options?

The trading of options, which only began in the early 1970's, has been increasing ever since with no sign of abating; recent growth has seen the derivatives market increasing ever since with no sign of abating; recently the market has increased from 220 trillion trades in the first half of 2004 to 370 trillion in the first half of 2006. It is not just stocks that are traded -




also commodities (oil, gold, tin, corn, etc.), exchange rates (Euro to Sterling etc.), etc.

Why is the options market such a big deal? Options appeal to two main types of investors - hedgers and speculators.

### 3.3.1 Options for hedging

This was how we introduced the idea of forward and option contracts. If a company or investor requires a certain amount of goods or currency in a certain amount of time then options provide insurance for cases where there are adverse market moves. It is sometimes possible to hedge against movements in a market which will affect your business. As an example, if jet fuel goes up then it costs British Airways more money to run their aircraft, so if they buy call options in jet fuel then if the price goes up then they make money to offset their operating losses. If the price has gone down then they're happy because their operating costs are low.

### 3.3.2 Options for speculating

If an investor has a hunch about which way a market is moving then he can obtain more leverage by using options. Consider the following example: an investor feels that Barclays is likely to increase in value over the next three months and has $5000 to invest. The current stock price is $20 and call options are available for three months at the cost of $1. Consider two possible alternatives, the stock price goes up to $35 or down to $15.

**P**rofit and loss from the two different strategies when speculating on Barclays stock

| | Stock price at expiry | |
|---|---|---|
| *Strategy* | $15 | $35 |
| Buy shares | -$1250 | $3750 |
| Buy call options | -$5000 | $45000 |

Investing the \$5000 in shares, enables the investor to buy 250 shares at \$20, so if the price drops to \$15 then the shares are now worth \$3750, realising a loss of \$1250, and if it increases to \$35 then they're worth \$8750, a profit of \$3750.

However, the options provide far more leverage in that a small increase or decrease in the underlying can realise big profits or losses. In this case the \$5000 buys 5000 call options with a strike price of \$25. If the price goes down to \$15 then none of these would be exercised and the investor would have lost their entire \$5000. If, on the other hand, the price has gone up to \$35 then each of the options enables the investor to make a \$10 profit, meaning they would make $5000 \times 10 = \$50000$ less the initial outlay of \$5000 which is a massive profit of \$45000.

# 4 Monte Carlo methods

## 4.1 Introduction

- We now look at a numerical scheme that uses the probabilistic solution - Monte Carlo techniques.
- The main idea behind the Monte Carlo technique is that you simulate paths that could be taken by the underlying asset (under the risk-neutral probability) and then use these to estimate an expected option price at expiry, which can be discounted back to today.
- Sadly, the convergence of Monte Carlo methods is slow and it is hard to determine the error terms.
- The convergence to the correct option value will be at a rate of $N^{-\frac{1}{2}}$ where $N$ is the number of sample paths.
- Practitioners love these methods!!

## 4.2 Large numbers

- If we have a sequence of independent, identically distributed random variables $Y_n$ then we have that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} Y_n = E[Y_1]$$

  which is the law of large numbers.
- In other words the expectation is exactly like taking a long run average (exactly as we'd expect), so to evaluate the expectation to any desired accuracy we can simply take more and more draws of $Y_n$.
- With the Monte Carlo technique what we are trying to do is to evaluate the value of $E[f(Y_T)]$ which is the expectation of a function of a random variable $Y_T$.

## 4.3 The time value of money

An amount, $A$, invested at a continually compounded rate of $r$ for $t$ years is worth $Ae^{rt}$. This time value of money is mainly used in its inverted form to determine what an expected amount in the future is worth today. This is known as **discounting**. For example, if an investor is going to receive \$100 at some time in the future $T$ then at an earlier time $t$ it is worth \$$100e^{-r(T-t)}$. Interest rates and discounting is used extensively in developing option pricing techniques.

## 4.4 Application to options

- If we consider $S_t$ as the value of a share price at time $t$, then the option value at expiry, $t = T$ we can think of as $V(S_T, T)$ and from the fundamental theorem of finance we know that

  $$V(S_t, t) = e^{-r(T-t)} E_t^Q[V(S_T, T)]$$

  if $r$ is constant, where $Q$ is the risk-neutral measure and $E_t$ denotes taking the expectation at time $t$. Note that the principle of *the time value of money* has been used here.

- Thus if we can estimate the expectation on the right hand side then we can simply discount this value at the risk-free rate to obtain the option price today. In fact, with Monte-Carlo methods it is also fairly straightforward to factor in stochastic interest rates as well.

## 4.5 Simple example with GBM

- If we assume that we are in the risk-neutral world and the underlying asset follows geometric Brownian motion, thus, under the real-world measure

  $$dS_t = \mu S_t dt + \sigma S_t dX$$

  and under the risk-neutral measure $\mu = r$ and so

  $$dS_t = rS_t dt + \sigma S_t dW$$

  where $W$ and $X$ are both Brownian motions under the respective measures.

- $\sigma$ is the *volatility* - a key parameter in the financial world - a measure of risk. Blue chip companies have small volatilities, start-up firms (for example) take on larger values. Typical values 0.2 - 0.4 per $(\text{annum})^{\frac{1}{2}}$.

- The above stochastic differential equation can be 'solved' exactly (there is quite a lot of theory behind this, involving a major branch of mathematics - Ito calculus) to yield:

  $$S_T = S_t \exp[(r - \tfrac{1}{2}\sigma^2)(T - t) + \sigma dW]$$

  or

$$S_T = S_t \exp[(r - \tfrac{1}{2}\sigma^2)(T - t) + \sigma\phi\sqrt{T - t}]$$

where $\phi$ here is a variable drawn at random from a Normal distribution with a mean of 0 and a variance of 1, $N(0,1)$.

- To estimate the expected option value at time $T$, $V(S_T)$ then we take random draws from the $N(0,1)$ distribution which enables us to calculate $S_T$ and then calculate $V(S_T)$. To get an approximation of the expectation we then average $V(S_T)$.
- Thus if the $n$th draw from the normal distribution gives $V^n(S_T)$ then by the law of large numbers:

$$\frac{1}{N}\sum_{n=1}^{N} V(S_T^n) \rightarrow E_t^Q[V(S_T)] \quad \text{as} \quad N \rightarrow \infty$$

- Now if we define the error from the $n$th sample path as $\epsilon_n$ so

$$\epsilon_n = V(S_T^n) - E_t^Q[V(S_T)]$$

### 4.6 Central limit theorem and error

- The Central Limit Theorem tells us that for large $N$ if the individual errors $\epsilon_n$ have variance $\nu = var(\epsilon_n)$ (which is the same for all $n$) then the error when approximating the expectation:

$$\frac{1}{N}\sum_{n=1}^{N} V(S_T^n) - E_t^Q[V(S_T)]$$

is approximately normally distributed with mean zero and variance $\nu/N$ and standard deviation $(\nu/N)^{\frac{1}{2}}$.

- Sadly this error bound is difficult to estimate as it is probabilistic, in that we only know the distribution of the errors rather than their actual values.
- Also the standard deviation of the error only declines with the square root of the number of paths $N$.
- For each individual path the error will be random, depending upon the draw of $\phi$.

### 4.7 The Monte-Carlo method for European options

- That gives the basics of the Monte Carlo method, it is very simple to implement for many different types of options.

- For a European call option the payoff at maturity $V(S_T)$ is given by

$$V(S_T) = \max(S_T - X, 0)$$

and so, to value the option one simulates $N$ possible values or paths for $S_T$ by making $N$ independent draws from $N(0,1)$ then to use these possible values, call them $\phi_n$ we have for $1 \le n \le N$

$$S_T^n = S_t \exp[(r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\phi_n\sqrt{T-t}]$$
$$f(S_t^n) = \max(S_t^n - X, 0)$$
$$V(S_t, t) = e^{-r(T-t)}\frac{1}{N}\sum_{n=1}^{N} V(S_t^n)$$

- The figure below shows a typical simulation for valuing a put option: $T = 1$, $S = X = 1$, $r = 0.05$, $\sigma = 0.2$.