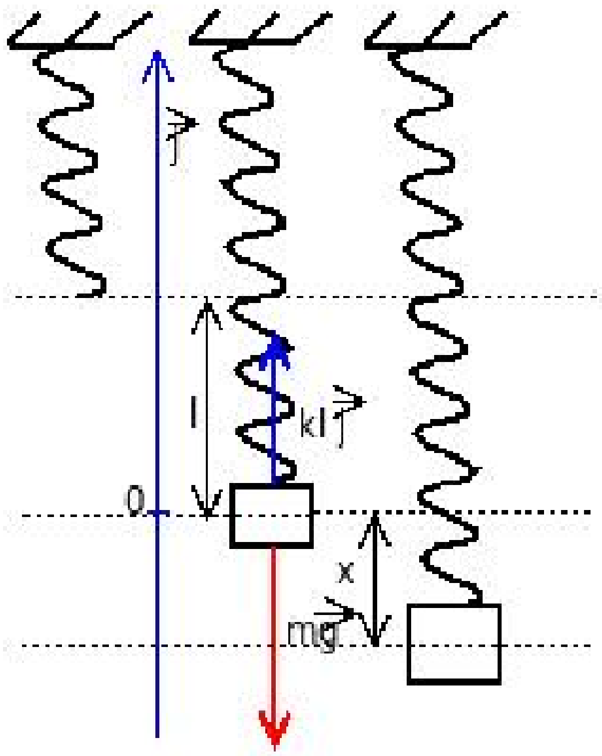
**Sheet 4 solutions: (produced by AIMS students '08)**
**The oscillating spring**

# 1 Free Oscillations



In the figure above, there exist three forces acting on the object namely:
1.The downward gravitational force: mg
2.The elastic restoring force in the spring: $-k(l+x)$
3.The damping force which is always proportional to the velocity as a result of the viscous fluid in which the object was immersed. It is always in the opposite direction: $-r\frac{dx}{dt}$.

Given the following parameters:
mass of object m $= 1kg$,
spring constant $k = 1N/m$,
damping constant $r = 2Nsec/m$.

(a) The equation governing the object's motion according to Newton's second law of motion is:

$$m\frac{d^2x}{dt^2} = mg - k(l+x) - r\frac{dx}{dt} \tag{1}$$

By Hooke's law, we know that $mg = kl$. Thus equation (1) becomes:

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0. \tag{2}$$

(b) To find the subsequent motion, we write the boundary and the initial conditions of (2): $x(0) = -0.25$, $\dot{x}(0) = 1$.
Substituting the values of m, r and k in (2) we have:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0 \tag{3}$$

we assume a solution of the form $x = e^{mt}$. By differentaiting and substituting into (3), we obtain

$$e^{mx}(m^2 + 2m + 1) = 0 \tag{4}$$

where $e^{mt} \neq 0$, hence

$$m^2 + 2m + 1 = 0$$

We obtain two equal roots $m_1 = m_2 = -1$. Thus, the general solution is:

$$X(t) = (A + Bt)e^{-t}. \tag{5}$$

Now applying the boundary and initial conditions, $x(0) = -0.25$, $v(0) = 1$, we obtain $A = -\frac{1}{4}$, $B = \frac{3}{4}$.
Hence the particular solution is given by

$$x_p(t) = \left(-\frac{1}{4} + \frac{3}{4}t\right)e^{-t} \tag{6}$$

which gives the subsequent motion of the object.

(c) To show that the object overshoot its equilibrium position, we consider the linear term in (5). The value of the constants A and B could be either positive or negative, therefore $x$ could shoot beyound zero.

(d) The sketch of position against time is shown in figure 1 below and gives a clearer picture of (c) above.

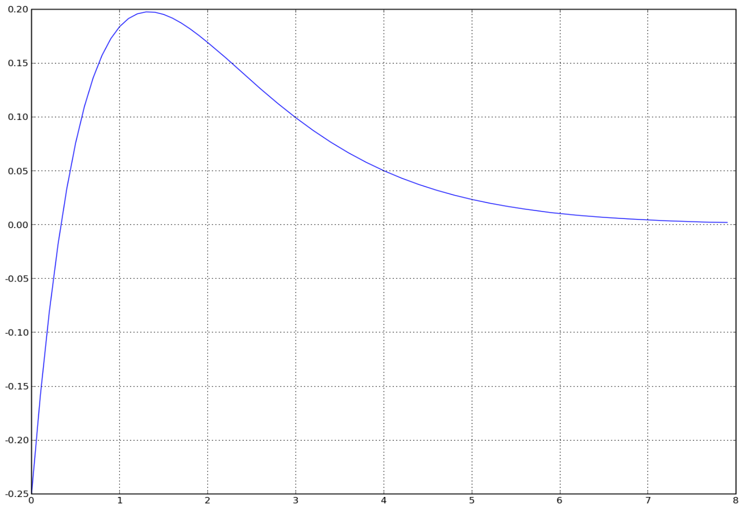


Figure 1: Position as a function of time

## 2 Superposition of forces

An object of mass 1kg is attached to a spring with spring constant 16 $Nm^{-1}$ and allowed to oscillate freely.

(a) Neglecting air resistance, we calculate the angular frequency $\omega$ and the period of the oscillations as below:
The equation of motion is given by :

$$\begin{aligned} & m\frac{d^2x}{dt^2} = mg - kx \qquad (7) \\ \Leftrightarrow\ & \frac{d^2x}{dt^2} = g - \frac{k}{m}x \\ \Leftrightarrow\ & \frac{d^2x}{dt^2} + \frac{k}{m}x = g \qquad (8) \end{aligned}$$

(i) $\omega$ is given by

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \quad (9) \\ &= \sqrt{\frac{16\text{Nm}_{-1}}{1\text{kg}}} \\ &= 4s^{-1} \end{aligned}$$

$$\boxed{\omega = 4s^{-1}} \quad (10)$$

(ii) The period $T$ will be

$$T = \frac{2\pi}{\omega} \quad (11)$$

$$\boxed{T = \frac{\pi}{2}} \quad (12)$$

(b) The object is acted upon by the external force $f_1 = cos2t$.

(i) The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = cos(2t) \text{ where } m = 1\text{kg and } k = 16\text{Nm}^{-1}$$

$$\boxed{m\frac{d^2x}{dt^2} + 16x = cos(2t)} \quad (13)$$

The homogeneous equation associated with the equation (13) is

$$\frac{d^2x}{dt^2} + 16x = 0 \quad (14)$$

with characteristic equation:

$$r^2 + 16 = 0 \Leftrightarrow r_1 = 4i,\ r_2 = -4i(Complexroots) \quad (15)$$

Thus solution to (14) is of the form

$$X_1(t) = Acos(4t) + Bsin(4t) \text{ (with A and B constants)} \quad (16)$$

We can rewrite (16) using trigonometry identity which reduces to (17)

$$X_1(t) = Rcos(4t - \phi) \text{ where } A = Rcos\phi \text{ and } B = Rsin\phi \quad (17)$$

Particular solution to (13)

Let $X_2(t) = Ce^{2it}$ be a solution to the equation (13). We have

$$\begin{aligned} \dot{X}_2(t) &= C2ie^{2it} && (18) \\ \ddot{X}_2(t) &= -4Ce^{2it} && (19) \end{aligned}$$

So (17) becomes:

$$\begin{aligned} C\left(-4e^{2it} + 16e^{2it}\right) &= e^{2it} \\ C\left(-4+16\right) &= e^{2it} \\ C &= \frac{1}{12} && (20) \\ & && (21) \end{aligned}$$

Considering the right-hand-side part of (13) which is a cosine function of t, $cos(2t)$ is the real part of the euler equation $e^{2i\omega t}$ so the real part gives the best comparison and we plug in the value of $C = \dfrac{1}{12}$

$$\begin{aligned} X_p(t) &= Re(X_2(t)) && (22) \\ X_p(t) &= \frac{1}{12}cos(2t) && (23) \end{aligned}$$

So the general solution is

$$\boxed{X(t) = Rcos(4t-\phi) + \frac{1}{12}cos(2t)} \qquad (24)$$

(c) The external force is changed to $f_2(t) = cos(4t)$

The equation of the motion becomes

$$\frac{d^2X}{dt^2} + 16X = cos(4t). \qquad (25)$$

Consider $X(t) = Cte^{4it}$ with the following first and second derivatives

$$\begin{aligned} \dot{X}(t) &= 4Ctie^{4it} + Ce^{4it} && (26) \\ \ddot{X}(t) &= 4Cie^{4it} - 16Cte^{4it} + 4Cie^{4it} && (27) \end{aligned}$$

Substituting into (25), we have

$$\begin{aligned}
8Cie^{4it} - 16Cte^{4it} + 16Cte^{4it} &= e^{4it} \\
8Cie^{4it} &= e^{4it} \\
C &= -\frac{i}{8} \qquad (28) \\
X(t) &= -\frac{i}{8}te^{4it} \qquad (29) \\
X_P(t) &= Re(X(t)) \qquad (30) \\
&= Re\left(-\frac{i}{8}(cos(4t) + isin(4t))\right)
\end{aligned}$$

$$\boxed{X_p(t) = \frac{t}{8}sin(4t)} \qquad (31)$$

(d) If both the forces $f_1$ and $f_2$ act on the object then the new equation of the motion becomes:

$$\frac{d^2X}{dt^2} + 16X = cos(4t) + cos(2t) \qquad (32)$$

We have the combination for the particular solution constituting both external forces $f_1(t)$ and $f_2(t)$ acting on the object as computed in (b) and (c) above

$$\boxed{X_p(t) = \frac{1}{12}cos2t + \frac{t}{8}sin(4t)} \qquad (33)$$

## 3 Resonance

An object of mass m = 4 kg is attached to a spring with spring constant $k$ = 64 N/m and is acted on by an externalforce $f(t) = Acos^3(pt)$ in the downwards direction. Ignoring air resistence lets find all values of $p$ at which resonance occurs.

Equation of the motion

$$\begin{aligned} m\frac{d^2X}{dt^2} &= mg - k(l+x) + f(t) \quad (34) \\ &= mg - kl - kx6 + f(t) \quad (35) \\ \frac{d^2X}{dt^2} &= -\omega^2 X + \frac{1}{m} f(t) \text{ where} \quad (36) \end{aligned}$$

$$\omega^2 = \frac{k}{m} = \frac{64k}{4} = 16s^{-1} \tag{37}$$

$$\frac{d^2X}{dt^2} = -16X + \frac{1}{4} f(t) \tag{38}$$

By Euler's formula, we get

$$\begin{aligned} cos^3(pt) &= \left(\frac{e^{ipt}+e^{-ipt}}{2}\right)^2 \left(\frac{e^{ipt}+e^{-ipt}}{2}\right) \quad (39) \\ &= \frac{1}{8}\left(e^{2ipt}+e^{-2ipt}+2\right)\left(e^{ipt}+e^{-ipt}\right) \\ &= \frac{1}{8}\left(e^{3ipt}+e^{ipt}+e^{-ipt}e^{-3ipt}+2e^{ipt}+2e^{-ipt}\right) \\ &= \frac{1}{8}\left(e^{3ipt}+e^{-3ipt}+3e^{ipt}+3e^{-ipt}\right) \\ &= \frac{1}{8}\left(2cos(3pt)+6cos(pt)\right) \quad (40) \end{aligned}$$

The equation of the motion becomes

$$\frac{d^2X}{dt^2} + 16X = \frac{A}{16}\left(cos(3pt) + 3cos(pt)\right). \tag{41}$$

The solution is of the form $X_1(t) = Ce^{3ipt}$ and $X_2(t) = De^{ipt}$
we have the derivatives given as

$$\dot{X_1}(t) = 3ipCe^{3ipt} \tag{42}$$

$$\ddot{X_1}(t) = -9p^2Ce^{3ipt} \tag{43}$$

$$\dot{X_2}(t) = ipDe^{ipt} \tag{44}$$

$$\ddot{X_2}(t) = -p^2De^{ipt} \tag{45}$$

Substituting into (41) and comparing the Euler equation equivalent of (41) with $\frac{A}{16}(e^{3ipt} + e^{ipt})$, we obtain the equations

$$-9Cp^2 + 16C = \frac{A}{16}, -Dp^2 + 16D = \frac{A}{16} \tag{46}$$

From which $C = \frac{A}{16}\left(\frac{1}{16-9p^2}\right)$ and $D = \frac{A}{16}\left(\frac{3}{16-p^2}\right)$

The steady state solution is gotten from the real part of the form of our solution assumed above as:

$$X_p(t) = Re\frac{A}{16}\left(\frac{1}{16-9p^2}e^{3ipt} + \frac{3}{16-p^2}e^{ipt}\right) \tag{47}$$

Therefore,

$$X_p(t) = \frac{A}{16}\left(\frac{1}{16-9p^2}cos(3pt) + \frac{A}{16}\frac{3}{16-p^2}cos(pt)\right) \tag{48}$$

which implies that there is resonance when

$$p = 4 \tag{49}$$

or

$$p = \frac{4}{3} \tag{50}$$

$$\omega_0 = 4 \tag{51}$$

# 4 Power and Driven Oscillations

Consider the figure below

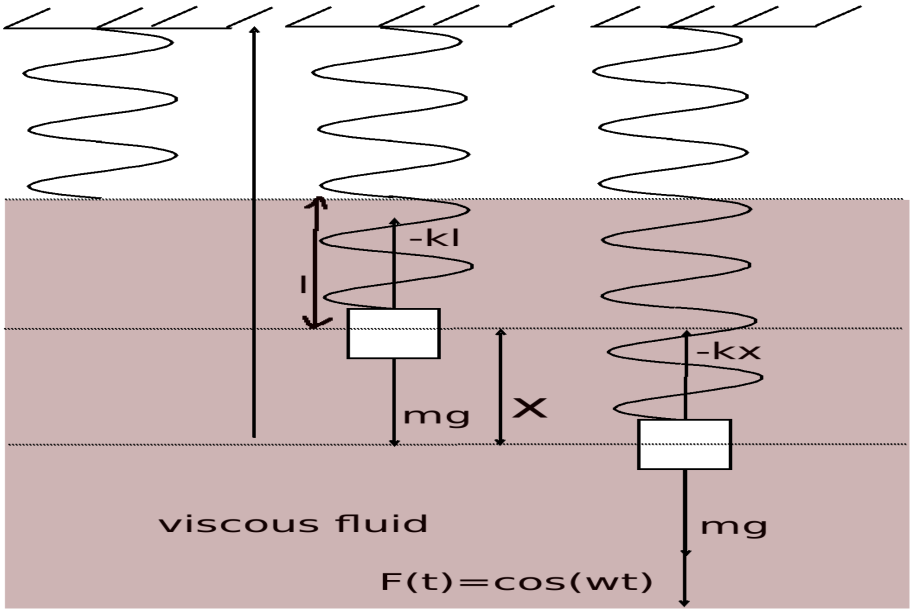
In this problem there are basically four forces acting on the object, namely:
1.The downward gravitational force: mg
2.The elastic restoring force in the spring: $-k(l+x)$
3.The damping force always proportional to the velocity which is as a result of the viscous substance in which the object was immersed, in the opposite direction: $-r\frac{dx}{dt}$
4.The external force, $f(t) = cos\omega t$.

The equation of motion for the object is given as:

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = cos\omega t \tag{52}$$

To deduce the rate of change of energy,
Given,

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{53}$$



differentiating (53) we have

$$\frac{dE}{dt} = m\ddot{x}\dot{x} + kx\dot{x} \tag{54}$$

Substituting for $m\dfrac{d^2x}{dt^2}$ from (52) in (54) we have $(cos\omega t - r\dot{x} - kx)\dot{x} + kx\dot{x}$ and we obtain

$$\frac{dE}{dt} = -r\dot{x}^2 + \dot{x}cos\omega t. \tag{55}$$

But $cos\omega t = f(t)$, thus (55) becomes

$$\frac{dE}{dt} = -r\dot{x}^2 + \dot{x}f \tag{56}$$

which is our desired rate of change of energy.

(b) From the equation of motion :

$$mx\ddot{x} + r\dot{x} = f(t) = \frac{f_0}{m}cos\omega t.$$

here

$$\frac{f_0}{m} = 1.$$

Considering the particular solution

$$x_p(t) = R(\omega)wcos(\omega t - \phi) \tag{57}$$

where $R(\omega)$ is the amplitude of the the displacement and $\phi = \frac{\Pi}{2}$ Given:

$$\begin{align}
\bar{P}(\omega) &= \frac{1}{T}\int_0^T f\dot{x}dt \tag{58}\\
&= \frac{1}{T}\int_0^T -R\omega sin(\omega t-\phi)cos\omega tdt \nonumber\\
&= \frac{-R\omega}{T}\int_0^T (sin\omega tcos\phi - sin\phi cos\omega t)cos\omega tdt \nonumber\\
&= \frac{-R\omega}{T}cos\phi\int_0^T sin\omega tcos\phi dt + \frac{R\omega}{T}sin\phi\int_0^T cos^2\omega tdt \nonumber\\
&= -\frac{R\omega}{T}\frac{cos\phi}{2}\int_0^T sin2\omega tdt + \frac{R\omega}{T}sin\phi\int_0^T 1-sin^2\omega tdt \nonumber\\
&= \frac{R\omega}{T}sin\left(T-\frac{T}{2}\right) \nonumber\\
\bar{P}(\omega) &= \frac{R\omega}{2}sin\phi \tag{59}
\end{align}$$

At resonance,
$\omega = \omega_0$ and so

$$R = \frac{1}{\sqrt{(\omega^2-\omega_0^2)+r^2w^2}} \longrightarrow \frac{1}{r\omega} \tag{60}$$

$$\begin{align}
tan\phi &= \frac{r\omega}{\omega^2-\omega^2} \tag{61}\\
tan\phi &= \infty \tag{62}
\end{align}$$

The value of $\phi$ for which tan is $\infty$ is $\frac{\pi}{2}$
Thus $sin\phi = 1$ and so from (59) we get our average power input per cycle

$$\boxed{\bar{P}(\omega)_{max} = \frac{1}{2r}}$$