Sheet 7: Phase diagrams

Differential equations **Aims 2010**

Hand in individual solutions to TWO questions out of S1, S2, S3 and group solutions to EITHER L1 OR L2 to your tutor by Friday 10 pm

S1 **Phase diagrams for one dimensional systems**

Draw phase diagrams for the equations

$$(i)\ \dot{x} = x - x^2, \qquad (ii)\ \dot{x} = \sin(x)$$

S2 **Stable points need not be asymptotically stable**

Show that the one dimensional system

$$\dot{x} = \begin{cases} -x^2 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad 0 \leq x \leq 1 \\ -(1-x)^2 & \text{if} \quad x > 1 \end{cases}$$

has many stable points which are not asymptotically stable.

L1 **Attracting points are stable in one dimension**

Consider the one dimensional autonomous system

$$\dot{x} = f(x),$$

where f is a continuously differentiable function. Show that for any solution $x(t)$

(a) If $f(x(t_0)) = 0$ for some t_0, then $f(x(t)) = 0$ for all $t \in \mathbb{R}$

(b) If $f(x(t_0)) > 0$ for some t_0, then $f(x(t)) > 0$ for all $t \in \mathbb{R}$. (The same holds for $>$ replaced with $<$)

(c) Use (a) and (b) to show that if an equilibrium point x^* is attracting then it must also be stable.

L2 **Two dimensional linear systems**

For each of the following systems

(i) Determine the type of the equilibrium point $\boldsymbol{x}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and state whether it is stable, asympotically stable or unstable.

(ii) Sketch the phase diagram for the system.

(iii) Draw graphs of the solutions x_1 and x_2 against t corresponding to the solution passing through the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$.

$$(a)\ \dot{\boldsymbol{x}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}, \quad (b)\ \dot{\boldsymbol{x}} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \boldsymbol{x} \quad (c)\ \dot{\boldsymbol{x}} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \boldsymbol{x}$$

$$(d)\ \dot{\boldsymbol{x}} = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{x} \quad (e)\ \dot{\boldsymbol{x}} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \boldsymbol{x}.$$

S3 **SIR Epidemics**

The SIR model is a classic model in epidemiology, and a significant improvement on the simple model discussed on sheet 1. It considers three subpopulations, the susceptibles S, the infectives I and the removed individuals R (meaning immune or dead or in quarantine). The susceptibles can become infective, and the infectives can become removed, but no other transitions are considered. Diagrammatically

$$S \to I \to R.$$

The total population $N = S + I + R$ remains constant. The model describes the movement between the classes by the system of differential equations

$$\frac{dS}{d\tau} = -\beta IS, \quad \frac{dI}{d\tau} = \beta IS - \gamma I, \quad \frac{dR}{d\tau} = \gamma I. \tag{1}$$

In terms of the fractions $x = S/N, y = I/N, z = R/N$, and the rescaled time variable $t = \gamma\tau$, the equations become

$$\frac{dx}{dt} = -R_0 xy, \qquad \frac{dy}{dt} = R_0 xy - y, \qquad \frac{dz}{dt} = y, \tag{2}$$

where $R_0 = \beta N/\gamma$ is an important parameter, called the reproductive ratio.

(a) Explain why x, y, z all lie in the interval $[0, 1]$ and why $x+y+z = 1$. Since the equations for x and y do not involve z we can concentrate on the x, y system, and compute z via $z = 1 - x - y$. Mark the region

$$D = \{(x, y) \in \mathbb{R}^2 | 0 \le x, y \le 1, x + y \le 1\}$$

in the xy plane.

(b) Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -R_0 xy \\ \frac{dy}{dt} &= R_0 xy - y. \end{aligned} \tag{3}$$

Deduce a differential equation for y as a function of x and find its general solution.

(c) Find all critical points of (3) in the region D. Considering $R_0 \le 1$ and $R_0 > 1$ separately, and using (a) if necessary, determine the nature of the critical points. Draw a phase diagram for each of the cases $R_0 \le 1$ and $R_0 > 1$.

d) Sketch the fractions y and x (infectives and susceptibles) as functions of t for trajectories starting near the disease free state $(x, y) = (1, 0)$. Hence show that the disease dies out if $R_0 \le 1$ but that an epidemic may occur if $R_0 > 1$. Also show that, regardless of the value of R_0, susceptibles remain in the population at the end of the disease. (*An epidemic is a disease which spreads rapidly and widely. In particular, the number of infectives increases during the initial period of an epidemic*)