Sheet 6: Oscillations

Differential equations **Aims 2010**

Hand in individual solutions to S1-S2 and a group solution to L by WEDNESDAY 11pm

S1 Cheap intercontinental travel

Imagine a tunnel drilled through the centre of the earth, from Spain to New Zealand. The total gravitational force exerted by the earth's matter on a capsule travelling through the tunnel can be computed according to the following rule: *if the capsule is a distance r away from the centre of the earth, the gravitational force it experiences is as if the earth's matter inside the radius r was concentrated at the earth's centre, and the matter outside the radius r was not there at all.*
(*i*) Find the mass M_E and radius R_E of the earth by typing "mass of earth" and "radius of earth" into google.
(*ii*) Assuming the earth to be a perfect ball with uniform mass density, compute the total mass of matter inside the ball of radius $r < R_E$ as as function of r.
(*iii*) Using the rule given *in italics* above, compute the gravitational force experienced by the capsule when it is a distance r away from the centre of the earth.
(*iv*) Derive the equation governing the fall of the capsule to the centre of the earth, neglecting air resistance and treating the capsule as a point particle.
(*v*) If the capsule is dropped into the tunnel in Spain with zero initial speed, find a formula for its position at time t seconds later.
(*vi*) Using the google calculator, or otherwise, compute the time it take the capsule to reach the centre of the earth (in minutes), and the capsule's speed (in km/h) at that point? How long does it take til the capsule is in New Zealand, what is the speed when it gets there, and what is its average speed for the journey from Spain to New Zealand?

S2 Resonance

An object of mass 4 kg is attached to a spring with spring constant 64 N/m and is acted on by an external force $f(t) = A\cos^3(pt)$ in the downwards direction. Ignoring air resistance, find all values of p at which resonance occurs.

L What was the problem with the Millenium bridge?

This mini-project is based on a real research article, which appeared the journal 'Nature' in 2005. I will make copies available. It's a short and well-written article which you will need to read for background information.

The mathematical model which is proposed in the Nature article consists of one real function $X(t)$ for the bridge's lateral displacement from the equilibrium postion, and angular coordinates Θ_i, $i = 1, \ldots, N$ to model the pedestrians: the angle Θ_i increases by 2π for a full left/right walking cycle by the i-the pedestrian. The functions $X(t)$ and $\Theta_i(t)$ satisfy a coupled set of ordinary differential equations. The bridge's displacement is affected by the pedestrians according to

$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N}\sin\Theta_i \qquad (1)$$

and the pedestrians' gait in turn is affected by the bridge according to

$$\frac{d\Theta_i}{dt} = \Omega_i + CA\sin(\Psi - \Theta_i + \alpha). \tag{2}$$

Here M, B, K, Ω_i, C and α are all real constants, but A and Ψ are related to the function X via

$$X = A\sin\Psi, \qquad \frac{dX}{dt} = A\sqrt{\frac{K}{M}}\cos\Psi, \tag{3}$$

The goal of the project is to solve this system of ODE's numerically for $N = 1$ and also for $N = 10$, $N = 100$ and $N = 200$, with physically reasonable values of M, B, K, Ω_i, C and α and initial conditions which you may choose. You should also plot X, Θ_i and the 'wobble amplitude'

$$A = \sqrt{X^2 + \frac{M}{K}\left(\frac{dX}{dt}\right)^2} \tag{4}$$

as a function of t.

We break this task down as follows:
(i) Simplify the model and solve the simplified model first! We consider only a single pedestrian, so $N = 1$, and we set $M = 1$, $K = 40$, $\Omega_1 = 6$. As a first step we consider the decoupled equations i.e. $G = C = 0$. We also set the damping to zero $B = 0$. Pick initial conditions (e.g., $X(0) = 1, dX/dt(0) = 0, \Theta_1(0) = 0$) and solve the resulting equation (1) for X and (2) for Θ_1 and plot both as a function of t. Also plot the wobble amplitude (4) as a function of t. Discuss your finding. Is it as you expect?
(ii) Repeat the above, but now use $B = 3$, then $B = 10$, $B = 12$ and $B = 15$. How does the behaviour of X and A change? Can you explain the behaviour?
(iii) Now we couple the equations, but still only allow one pedestrian. We set $G = 30, C = 16$. Then expand the right hand side of (2) using an appropriate trigonometric identity and (3) to replace $A\sin\Psi$ and $A\cos\Psi$. Show that (2) becomes

$$\frac{d\Theta_i}{dt} = \Omega_i + C\left(X\cos(\alpha - \Theta_i) + \sqrt{\frac{M}{K}}\frac{dX}{dt}\sin(\alpha - \Theta_i)\right). \tag{5}$$

Now solve the coupled system (1) and (5), still keeping $N = 1$. Use several sets of parameter values for $M, B, K, \Omega_1, \alpha$. In each case plot X, A and Θ_1 as a function of t.
(iv) (OPTIONAL) Finally consider a crowd of pedestrians on the Millenium bridge: move up to $N = 10, N = 100$ and $N = 200$. In each case you will need to generate a random set of gait frequencies $\Omega_i, i = 1, \ldots, N$, with a realistic distribution and mean. You can take the following parameters for the north span of the bridge (fom the paper ' The Millennium Bridge London: problems and solutions' by P. Dallard, A. J. Fitzpatrick, A. Flint,S. Le Bourva, A. Low, R. M. R. Smith and M. Willford, in The Structural Engineer, Vol 79 No 22 (2001), 17—33)

$$M \approx 113 \times 10^3\text{kg}$$
$$8.78 \times 10^3\text{kg/s} \le B \le 11.7 \times 10^3\text{kg/s}$$
$$K \approx 4730 \times 10^3\text{kg/s}^2$$

Reasonable values for the remaining parameters are $G \approx 30$ Newton, $C \approx 16\ \text{m}^{-1}\text{s}^{-1}$. Now solve the system of ODEs for these parameter values, with initial conditions of your choice. Can you reproduce the behaviour shown in the Nature article?