A seminar on the Picard-Lindelöf theorem

Differential equations **AIMS 2010**

1 Contraction mapping theorem

Remind yourself of the definitions of a **norm**, **metric space**, **Cauchy sequence** and **completeness** of a metric space. Recall that every normed space with norm $||\cdot||$ is automatically a metric space, with metric d defined via

$$d(x, y) = ||x - y||.$$

Definition 1.1. *A Banach space is a normed, complete vector space over* $\mathbb{R}$ *or* $\mathbb{C}$.

Exercise 1.2. (i) *Show that* $||x|| = max(|x_1|, \ldots, |x_n|)$ *defines a norm on* $\mathbb{R}^n$ *and that, with that norm,* $\mathbb{R}^n$ *is a Banach space.*
(ii) *Let* K *be a compact subset of* $\mathbb{R}$ *- for example, a closed and bounded interval. Show that the set* $C(K, \mathbb{R}^n)$ *of continuous functions from* $\mathbb{R}$ *to* $\mathbb{R}^n$ *with supremum norm*

$$||x||_\infty = sup_{t\in K}||x(t)||.$$

is a Banach space.

One can show that on $\mathbb{R}^n$ all norms are equivalent (in particular, they define the same notions of continuity and convergence). In the following we write $|| \cdot ||$ for a norm on $\mathbb{R}^n$. The arguments we give are valid regardless of the norm we choose.

Definition 1.3. *A map* $F : M \to M$ *from a metric space* (M, d) *to itself is called a contraction map if there exists a* $k < 1$ *such that* $\forall x, y \in M$,

$$d(F(x), F(y) \leq kd(x, y).$$

Exercise 1.4. *Consider the Banach space* $V = \mathbb{R}$. *Give examples of maps* $F : \mathbb{R} \to \mathbb{R}$ *which are contraction maps. Also give examples of maps which are only defined on some subset* $\Omega \subset \mathbb{R}$ *and which are contraction maps* $F : \Omega \to \Omega$.

Theorem 1.5. *(***Contraction Mapping Theorem***) Let* Ω *be a closed subset of a Banach space* V *and* $F : \Omega \to \Omega$ *be a contraction map. Then there exists a unique* $x^* \in \Omega$ *such that* $F(x^*) = x^*$. *The element* x^* *is called a fixed point of the contraction map* F.

The idea of the proof is to start with an arbitrary element $x_0 \in \Omega$ and to consider the sequence x_n defined via

$$x_{n+1} = F(x_n), \qquad n = 0, 1, 2 \ldots.$$

One shows that (x_n) is a Cauchy sequence and, since Ω is a closed subset of a Banach space, this Cauchy sequence has a limit $x^* \in \Omega$ which satisfies $F(x^*) = x^*$ (the latter can be deduced from the uniqueness of the limit).

Exercise 1.6. *Consider again the Banach space* $V = \mathbb{R}$. *Consider the function* $F(x) = e^{-x}$ *and show that it is a contraction map* $[0, 1] \to [0, 1]$. *Sketch a graph of the function* F *together with a graph of the identity map* $id(x) = x$. *Explain why the intersection of the two graphs is the fixed point* x^* *and draw a picture of the sequence* x_n, *starting with an arbitrary* x_0. *Give a geometrical argument why this sequence converges to* x^* *if* F *is a contraction map.*

2 Strategy of proof

As shown in the lectures, the initial value problem

$$\frac{dx}{dt} = f(t, x), \qquad x(t_0) = x_0 \tag{2.1}$$

for a vector-valued differentiable function defined on some interval I containing t_0

$$x : I \subset \mathbb{R} \to \mathbb{R}^n$$

is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

This integral equation makes sense for any continuous function $x : I \to \mathbb{R}^n$. The idea behind the proof of the Picard-Lindelöf theorem is to show that for a suitable interval $\tilde{I} \subset I$, and after restricting the range of x to a suitable region $E \in \mathbb{R}^n$ (containing x_0) the map

$$\Gamma : C(\tilde{I}, E) \to C(\tilde{I}, E), \qquad \Gamma(x)\,(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{2.2}$$

is a contraction mapping. Its unique fixed point is then the unique solution of the initival value problem (2.1).
In order to apply the fixed point theorem, we need

(a) to choose $\tilde{I}$ and E so that $C(\tilde{I}, E)$ is a closed subset Ω of a Banach space,

(b) to adjust the choices of $\tilde{I}$ and E, if necessary, to make sure that Γ maps Ω into Ω,

(c) To impose suitable conditions to ensure that $\Gamma : \Omega \to \Omega$ is a contraction mapping.

There are different versions of the Picard-Lindelöf theorem in which these choices are made in different ways. However, in all versions, it is required to impose a condition on the function f, called the Lipschitz condition:

Definition 2.1. *A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to satisfy a Lipschitz condition in the set $U \in \mathbb{R}^m$ if there exists a real, positive constant k so that $||f(x) - f(y)|| \leq k||x - y||$ for $x, y \in U$. In that case we also say that the function is Lipschitz continuous.*

Exercise 2.2. (*i*) *Which of the following functions are Lipschitz continuous?*
(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$, *(b)* $f : [1, 6] \to \mathbb{R}, f(x) = x^2$, *(c)* $f : (0, 1) \to \mathbb{R}, f(x) = 1/(1 - x)$
(*ii*) *Show that Lipschitz continuous functions are continuous. Show also that, for a cube $E = \{\xi \in \mathbb{R}^n | \, ||\xi - x_0|| \leq d\}$ any continuously differentiable function $f : E \to \mathbb{R}^n$ is Lipschitz continuous.*

In the next two sections we will prove two versions of the Picard-Lindelöf theorem:

(G) The global version: we let $I = \mathbb{R}$, $E = \mathbb{R}^n$, and require f to satisfy a global Lipschitz condition. Then we can prove the unique existence of a global solution.

(L) The local version: allow I and E to be suitable subsets of $\mathbb{R}$ and $\mathbb{R}^n$, and require only that f satisfies a Lipschitz condition in those subsets. Then we can prove the unique existence of a solution in an interval (local solution).

3 Global version

Theorem 3.1. *Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function satisfying a Lipschitz condition in x, with Lipschitz constant independent of t, i.e.*

$$\exists K > 0 \textit{ such that } \forall t \in \mathbb{R}\ \forall x, y \in \mathbb{R}^n \quad ||f(t,x) - f(t,y)|| \leq K||x - y||.$$

Then the initial value problem (2.1) *has a unique solution which is defined everywhere on $\mathbb{R}$.*

4 Local version

Theorem 4.1. *Let $I = [t_0 - T, t_0 + T]$ and $E = \{\xi \in \mathbb{R}^n |\, ||\xi - x_0|| \leq d\}$. Assume that $f : I \times E \to \mathbb{R}^n$ is a continuous function satisfying a Lipschitz condition in $x \in \mathbb{R}^n$, with Lipschitz constant independent of t, i.e.*

$$\exists K > 0 \textit{ such that } \forall t \in I\ \ \forall x, y \in E \quad ||f(t,x) - f(t,y)|| \leq K||x - y||.$$

Then there exists a $\delta > 0$ so that initial value problem (2.1) *has a unique solution in the interval $[t_0 - \delta, t_0 + \delta]$.*

5 Maximal solutions

The local version of the Picard-Lindelöf theorem does not specify the size of the interval in which the unique solution exists. In particular, one would like to say something about the largest possible interval in which the solution exists. This is the issue we address in this final section.