School of Computing and Mathematics

Probability Crash Course: Introductory Concepts

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Overview: This webfile is designed as a revision aid to some introductory concepts in probability. It is intended to supplement a formal encouter with a text book or a set of lectures. These notes are meant to be slightly interactive, mysterious green dots, squares and boxes appear which you can click on to answer questions and check solutions.

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Aims

- To explain the concepts “experiment”, “sample space”, “event” (composite, simple, event space)
- To introduce the Venn diagram
- To introduce tree diagrams
- To introduce elementary set operations
- To introduce the Axioms of probability

1. Definitions

An *experiment* is any process that generates a set of outcomes where the outcome is uncertain. For a random experiment:

- The *sample space* Ω is the set of all possible outcomes
- An *event* is a subset of the sample space
  - A *simple event* is an event which cannot be a union of other events
  - A *composite event* is an event which is not a simple event
- The *event space* is the set of all events (evento di spazio)

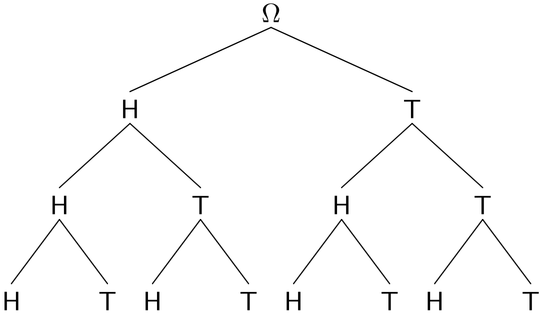
1.1. Example

Consider an experiment in which three coins are tossed. We are interested in the *event* that all coins will show the same face.

1in

The following tree diagram denotes the breakdown of the sample space (Ω). After the first coin toss we have either a head or a tail. After the second, for either of the possible outcomes of the first toss we also have a head or tail. Finally, after the third coin toss, we have a row denoting that a head or tail is possible for any of the previous outcomes.

1.2. Tree diagram



In total therefore, the tree diagram shows on the bottom row that there are a total of 2^3 possible outcomes in the sample space. In other words, there are 8 different ways of travelling though the tree diagram, resulting in the following outcomes:

- HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

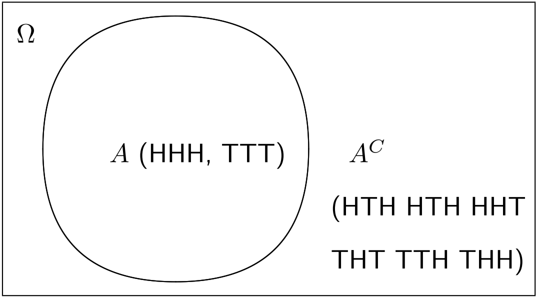
We are interested in the event that all faces are the same. There are two outcomes in the sample space which correspond to this event - denoted in red.

Recap:

- Sample Space (Ω) - the eight outcomes for the experiment:
  HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.
- Event (A) - the two outcomes of interest: HHH, TTT

There are 8 outcomes in our sample space, two of them correspond to the event "all three coins show the same space". It is conventional to depict this by a Venn diagram:

1.3. Venn diagram



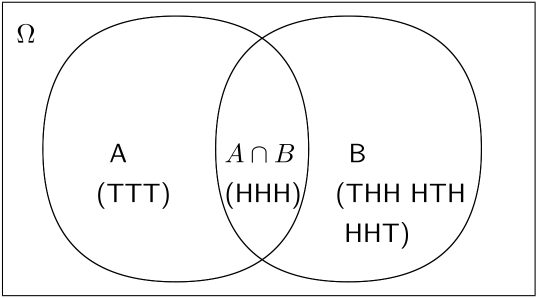
In this diagram, A denotes the event "all three faces show the same", and A^C denotes "A complement" (also known as "not A"), in other words the six outcomes where the faces are not identical.

2. Multiple events

For the same experiment, consider again the event A, "all three coins show the same face". But now, consider a second event, B, which we define as "two or more coins show a head"

- A: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.
- B: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

As before, 2 out the 8 outcomes correspond to event A. We can also see that 4 of the 8 outcomes correspond to event B. We can also see that one outcome (HHH) corresponds to both event A and event B. We can extend the Venn diagram to illustrate this:



You can see that we use the Venn diagram to visually illustrate that the outcome HHH is in both event A and event B. Standard set theory notation $A \cap B$ is used to denote that this is an “intersection”.

- Which outcomes are in $(A \cup B)^C$?

2.1. Set theory notation

For completeness, we quote a table from Grimmett and Stirzaker (2004) which gives a full listing of set theory and probability theory notation.

Notation	Set theory	Probability Theory
Ω	A collection of objects	Sample space
ω	A subset of Ω	Elementary event/outcome
A	A subset of Ω	Event that some condition A occurs
A^C	Complement of A	No event A occurs
$A \cap B$	Intersection of A and B	Both A and B occur
$A \cup B$	Union of A and B	Either A occurs, or B occurs or both
$A \setminus B$	Difference	A occurs but B does not occur
$A \triangle B$	Symmetric difference	Either A or B but not both
$A \subseteq B$	Inclusion	If A occurs then B occurs
$\emptyset$	Empty set	Impossible event
Ω	Whole space	Certain event

We should be able to verify standard operations such as:

- $A \cap B$: HHH
- $A \cup B$: HHH, TTT, HTH, HHT, THH
- B^C: TTT, TTH, THT, HTT
- $(A \cup B)^C$: TTH, THT, HTT

John Venn has in interesting family background, see MacTutor article on John Venn.

2.2. Commutation and association

We can extend these operations beyond two events, for example:

- $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$
- $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$

whereas:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

If we are happy with this, we can see how these can be used as the basis for probability.

3. Probability

Consider a random experiment with a finite number of outcomes, each with the same probability. For event A, it is clear that:

$$\text{Probability of an event} = \frac{\text{Number of possible sample points consistent with this event}}{\text{Total number of sample points}}$$

Or in a better notation:

$$p[A] = \frac{n[A]}{n[\Omega]} \tag{1}$$

where A denotes the event we are interested in, $n[A]$ is the number of ways in which A can happen and $n[\Omega]$ is the sample space.

Recall from our first Venn diagram, where we used A^C to denote the *complement* of A (i.e. those outcomes that weren't in A), then the probability of A^C (probability that A doesn't happen), $p[A^C]$ is:

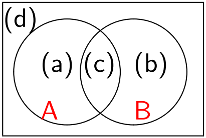
$$p[A^C] = 1 - p[A]$$

Likewise, if we consider the second Venn diagram, and consider that $p[B]$ denotes the probability of event B happening we have:

- $P[A \cup B]$: probability that event A or event B, or both happens
- $P[A \cap B]$: probability that both A and B happen

You need to click the Begin Quiz before you start, and the End Quiz box when you finish (you will then be told the correct answers).

Begin Quiz Identify the areas (to be divided by $n[\Omega]$) on the Venn diagram which correspond to the following events:

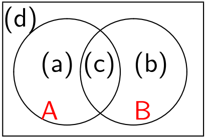


1. $p[A \cap B]$

(a) (a)
(b) (b)
(c) (c)
(d) (d)
(e) (a) and (b)
(f) (a) and (c)
(g) (b) and (c)
(h) (b) and (d)
(i) (a) and (d)
(j) (a),(b),(c)
(k) (a),(b),(d)
(l) (b),(c),(d)

2. $p[A^C]$

(a) (a)
(b) (b)
(c) (c)
(d) (d)
(e) (a) and (b)
(f) (a) and (c)
(g) (b) and (c)
(h) (b) and (d)
(i) (a) and (d)
(j) (a),(b),(c)
(k) (a),(b),(d)
(l) (b),(c),(d)

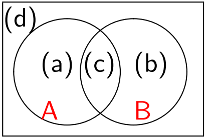


3. $p[A^C \cap B]$

(a) (a)
(b) (b)
(c) (c)
(d) (d)
(e) (a) and (b)
(f) (a) and (c)
(g) (b) and (c)
(h) (b) and (d)
(i) (a) and (d)
(j) (a),(b),(c)
(k) (a),(b),(d)
(l) (b),(c),(d)

4. $p[A^C \cup B^C]$

(a) (a)
(b) (b)
(c) (c)
(d) (d)
(e) (a) and (b)
(f) (a) and (c)
(g) (b) and (c)
(h) (b) and (d)
(i) (a) and (d)
(j) (a),(b),(c)
(k) (a),(b),(d)
(l) (a),(b),(c),(d)

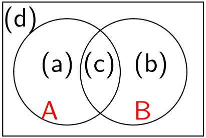


5. $p[A \cup B]$

(a) (a) (b) (b) (c) (c) (d) (d)
(e) (a) and (b) (f) (a) and (c) (g) (b) and (c) (h) (b) and (d)
(i) (a) and (d) (j) (a),(b),(c) (k) (a),(b),(d) (l) (b),(c),(d)

6. $p[A^C \cup B]$

(a) (a) (b) (b) (c) (c) (d) (d)
(e) (a) and (b) (f) (a) and (c) (g) (b) and (c) (h) (b) and (d)
(i) (a) and (d) (j) (a),(b),(c) (k) (a),(b),(d) (l) (b),(c),(d)



7. $p[A \cup B^C]$

(a) (a)
(b) (b)
(c) (c)
(d) (d)
(e) (a) and (b)
(f) (a) and (c)
(g) (b) and (c)
(h) (b) and (d)
(i) (a) and (d)
(j) (a),(b),(c)
(k) (a),(c),(d)
(l) (b),(c),(d)

8. $p[A^C \cap B]$

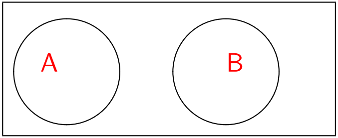
(a) (a)
(b) (b)
(c) (c)
(d) (d)
(e) (a) and (b)
(f) (a) and (c)
(g) (b) and (c)
(h) (b) and (d)
(i) (a) and (d)
(j) (a),(b),(c)
(k) (a),(b),(d)
(l) (b),(c),(d)

End Quiz Correct Points: Score:

3.1. Some other definitions

- **Mutually exclusive events**

These are events which cannot happen together:

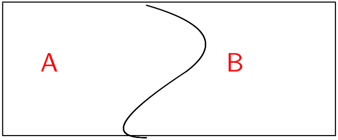


This implies:

- $p[A \cap B] = 0$
- $p[A \cup B] = p[A] + p[B]$

(a trivial example from our three coin problem would be the event "all three heads" and the event "all three tails")

- **Exhaustive / Complementary events**



Which implies:

- $p[A] + p[B] = 1$

For later use, please note that we could also say that B is A^C and hence:

- $A \cup A^C = \Omega$
- $p[A] + p[A^C] = 1$
- $p[A^C] = 1 - p[A]$

3.2. Axioms of probability

The system of probability we have been working through took a long time to develop, and wasn't fully formalised until the 1930's by Kolmogorov (see www.gap-system.org/ history/Biographies/Kolmogorov.html). He stated three fundamental Axioms, from which all other results can be derived.

- $p[A] \geq 0$ for any event A
- $p[\Omega] = 1$ where Ω is the sample space
- If $[A_i]; i = 1, 2, \ldots$ are mutually exclusive then $p[A_1] \cup p[A_2] \cup \ldots = p[A_1] + p[A_2] + \ldots$

Mutually exclusive means that $A_i \cap A_j = \emptyset$ for all $i \neq j$

We shall examine the implications of these Axioms in terms of obtaining mathematical functions that can serve as models for probability (probability distribution functions) in week 2. The intuitive consequences of these results are:

- (Non-negative) Probability can never be negative
- (Total probability) The probability of a sample space must equal 1
- (Countable additivity) The probability of observing two (or more) mutually exclusive events is the sum of their individual probabilities

However, we need to derive some additional results from these Axioms in order to be able to carry out useful probability calculations.

3.3. The addition rule

The third Axiom tells us that for events which are mutually exclusive, the addition rule is given by:

$$p[A \cup B] = p[A] + p[B]$$

However, we have already seen an example (the three coin problem) where events are not mutually exclusive - the outcome HHH was in both event A and event B. In this case, when events are not mutually exclusive, the addition rule is:

$$p[A \cup B] = p[A] + p[B] - p[A \cap B]$$

To verify this, draw the Venn Diagram. If we merely added $p[A]$ to $p[B]$ for events with an intersection, we would add the probability corresponding to $p[A \cap B]$ twice, and hence we need to subtract one of these areas.

What does this tell us about the value of $p[A \cap B]$ for for mutually exclusive events?

3.4. Independent events

Two events can be considered independent if they have no effect on each other. We would assume our coin tosses were independent - the second coin can't "know" whether the first coin landed heads or tails and hence it's probability of landing heads or tails is unaffected by the first throw. We consider conditional probability further in week 2, but for now we wish only to note that:

Two events can be considered independent when:

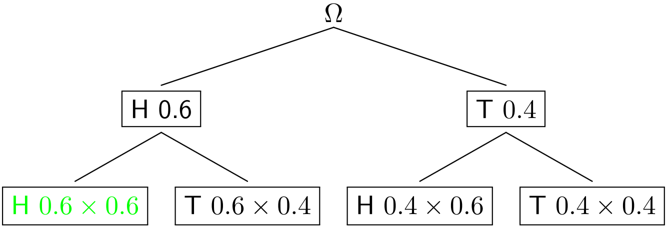
$$p[A \cap B] = p[A]p[B]$$

For example, for unbiased coins, $p[\text{Coin 1 shows H}] = 0.5$ and $p[\text{Coin 2 shows H}] = 0.5$, hence:

$$p[\text{Coin 1 shows H and Coin 2 shows H}] = 0.5 \times 0.5 = 0.25$$

This can be extended to more than two events.

It can also be used to extend our tree diagram examples. What if we have biased coins, where $p[\text{Coin shows H}] = 0.6$.



For example, we can see with such a biased coin that

$$p[\text{Both coins show H}] = 0.6 \times 0.6 = 0.36$$

which is somewhat higher than the 0.25 we would expect from unbiased coins.

3.5. Summary

We have reviewed the basic tools of probability theory. Please note:

- *Mutually exclusive (disjoint)* events are events that can't happen together, *independent* events are events that can (and do) happen together but in such a way that one event has no influence over the other. Sometimes the linguistics makes these concepts a little unclear.
- We have not yet considered conditional probability. We shall do that next week, which will lead us into direct application of Bayes theorem.
- To date, we have been talking about probability in the sense of a long run frequency (if we tossed a coin 1,000 times, we would expect very close to 500 heads). We shall next contrast this concept with "subjective" measures of probability.

First, we should consolidate our understanding of these introductory topics. Please ask for further problem sheets.

3.6. Exercises

There are six self marking electronic quizzes. For each of these, you need to click the Begin Quiz before you start, and the End Quiz box when you finish. There are then some paper exercises - feel free to ask for further problem sheets if you wish.

- **Exercise P.1**

Begin Quiz Consider that A and B are *mutually exclusive* events. It is known that $p[A] = 0.2$ while $p[B] = 0.3$.

1. Find $p[A^C] =$ ☐
2. Find $p[B^C] =$ ☐
3. Find $p[A \cup B] =$ ☐
4. Find $p[A \cap B] =$ ☐
5. Find $p[A^C \cap B^C] =$ ☐

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; Shift ⇑ + click to get the solution.

- **Exercise P.2**

Begin Quiz Now consider that A and B are *not mutually exclusive* events. It is known that $p[A] = 0.2$ while $p[B] = 0.3$ and also that $p[A \cap B] = 0.1$.

1. Find $p[A^C] =$
2. Find $p[B^C] =$
3. Find $p[A \cup B] =$
4. Find $p[A \cap B] =$
5. Find $p[A^C \cap B^C] =$

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; shift + click to get the solution.

Exercise P.3

Begin Quiz *Please give all answers to 4 decimal places*

Consider an office worker who has access to two PCs. These PCs work independently, and have a probability $\pi = 0.95$ of working correctly when switched on in the morning. What is the probability on a randomly chosen morning[1]:

1. Both PCs work $=$ ☐
2. At least one of the PCs works $=$ ☐
3. Exactly one of the PCs works: ☐

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; shift + click to get the solution.

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[1] You might need to draw a tree diagram

- **Exercise P.4**

Begin Quiz *Please give all answers to 2 decimal places*

Consider mass produced hinges. These hinges suffer from two kinds of fault. Fault A is found in 10% of hinges, fault B is found in 15% of hinges. Both faults are found in 8% of hinges. A single hinge is selected at random. To 2 decimal places, find the probability that:

1. It is faulty: ☐

2. It has fault A alone: ☐

3. It has neither fault: ☐

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; shift + click to get the solution.

• Exercise P.5

Begin Quiz *Please give all answers to 2 decimal places*

Consider two *mutually exclusive* events A and B. We know that $p[A] = 0.33$ and $p[B] = 0.42$. Find:

1. $p[A^C]$

2. $p[B^C]$

3. $p[A \cap B]$:

4. $p[A \cup B]$:

5. $p[A \cap B^C]$:

6. $p[A^C \cap B^C]$:

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; shift + click to get the solution.

Exercise P.6

Begin Quiz *Please give all answers to 2 decimal places*

Consider a married couple living in a certain suburb, who have been invited to participate in an election o a school board. The probability that the man will vote is 0.21, the probability that the woman will vote is 0.28 and the probability that both will vote is 0.15. Find the probability that:

1. At least one will vote: ☐
2. Only one of them will vote: ☐

End Quiz Correct Points: Score:

Click on the Ans box to get the correct answer; shift + click to get the solution.

Paper exercises

Here are some exercises for discussion.

1. You have a *fair* six sided die. You throw the die three times. What is the probability that you get
    - a 1 on the first throw, a 2 on the second throw and a 3 on the third throw
    - a 1, 2 and a 3 in any order
    - three 6s
    - one 4 and two 5s in any order
2.
    - What is the probability of getting the sequence $HHTT$ in four tosses of a fair coin.
    - Given you have already had a HH in your four coin toss experiment, what is the probability that the remaining two throws will yield TT
    - If you do not know the first two throws, what is the probability that the remaining two throws will yield TT

3. Box 1 has 1 red ball and 3 black balls. Box 2 has 1 red ball, 1 white ball and 1 black ball. Box 3 has 1 red ball and 1 black ball.

    - A box is chosen at random, and a single ball is chosen. What is the probability that it is red.
    - If a red ball is chosen, what is the probability that we had selected box 1, box 2 or box 3?

Please ask if you would like further problem sheets

Solutions to Quizzes

Solution to Quiz: The probability that "not A" happens, given that $p[A] = 0.2$ is $1 - p[A] = 0.8$.

Click on that green button to return to the quiz → 🟩

Solution to Quiz: The probability that "not b" happens, given that $p[B] = 0.3$ is $1 - p[A] = 0.7$.

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Solution to Quiz: The probability that either A or B happens, given that $p[A] = 0.2$, $p[B] = 0.3$ and that they are *mutually exclusive events* is $p[A] = p[B] = 0.5$.

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Solution to Quiz: The probability that both A and B happen, given that they are mutually exclusive events is 0 by definition.

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Solution to Quiz: Note that $p[A^C] = 0.8$ and contains $p[B] = 0.3$; likewise $p[B^C] = 0.7$ and contains $p[A] = 0.2$

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Solution to Quiz: The probability that "not A" happens, given that $p[A] = 0.2$ is $1 - p[A] = 0.8$.

Click on that green button to return to the quiz → 🟩

Solution to Quiz: The probability that “not b” happens, given that $p[B] = 0.3$ is $1 - p[B] = 0.7$.

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Solution to Quiz: The probability that either A or B happens, given that $p[A] = 0.2$, $p[B] = 0.3$ and that $p[A \cap B] = 0.1$ is given by $p[A] + p[B] - p[A \cap B] = 0.4$.

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Solution to Quiz: The probability that both A and B happen is 0.1 according to the question definition.

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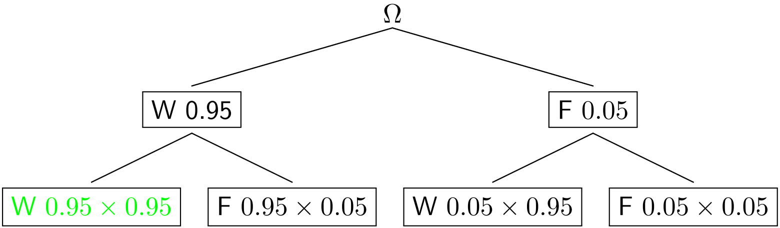
Solution to Quiz: Remember we now have $p[A \cap B] = 0.1$

Click on that green button to return to the quiz → 🟩



Solution to Quiz: Denoting a working PC by W, and a failed PC by F, we have one outcome where both PCs work, denoted in green, hence a probability of 0.9025.

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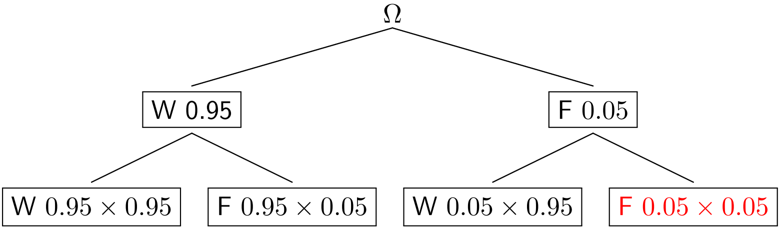

Solution to Quiz: One way of solving this is to consider that we have only one event whereby the condition is not met (coloured red); we want the probability of $1 - (none)$.

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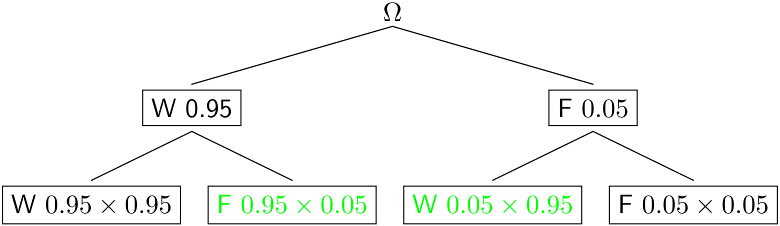

Solution to Quiz:

There are two events where exactly one PC is working, we therefore need to know $2 \times (0.05 \times 0.95)$.

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Solution to Quiz: We want $p[A] \cup p[B] = p[A] = p[B] - p[A \cap B] = 0.1 + 0.15 - 0.08 = 0.17$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: We know that $p[A] = 0.1$ and $p[A \cap B] = 0.08$, so $p[A] - p[A \cap B] = 0.02$

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Solution to Quiz: Here we want $1 - p[A \cup B]$, the latter value was given earlier so the answer here is $1 - 0.17$.

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Solution to Quiz: We want $1 - p[A]$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: We want $1 - p[B]$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: As these are mutually exclusive, $p[A \cap B] = 0$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: As these are mutually exclusive, as noted in the earlier question $p[A \cap B] = 0$, so here we want $p[A] + p[B] - 0$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: As these are mutually exclusive, we want the intersection of A and everything not in B, which is A.

Click on that green button to return to the quiz → 🟩

Solution to Quiz: As these are mutually exclusive, we want $1-(p[A]+p[B]) = 0.25$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: We want $p[A \cup B] = (0.21 + 0.28) = 0.34$

Click on that green button to return to the quiz → 🟩

Solution to Quiz: We know that $p[A \cup B] = 0.34$ from the previous question. To solve this one, we need $p[A \cup B] - p[A \cap B] = 0.34 - 0.15 = 0.19$

Click on that green button to return to the quiz → ■