School of Mathematics and Statistics

Probability Crash Course: Important Discrete distributions

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Overview: This webfile is designed as a revision aid to some introductory concepts in probability. It is intended to supplement a formal encouter with a text book or a set of lectures. These notes are meant to be slightly interactive, mysterious green dots, squares and boxes appear which you can click on to answer questions and check solutions.

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1. Some important examples of discrete distributions

1.1. Bernoulli trials

By convention we write $X \sim Bern(\pi)$

Repeated *independent* trials are called Bernoulli if:

- There are two possible outcomes (conventionally denoted success and failure)
- The probabilities, (p and $q = 1 - p$) remain the same from trial to trial

An example of a Bernoulli trial is a coin tossing experiment. Also, if we have a stable production process the probability of selecting a defective item might be Bernoulli. Can you suggest some other examples?

If we denote the parameter by π, we have:

$$\begin{cases} p[Z=1] & = & \pi \\ p[Z=0] & = & 1-\pi \end{cases}$$

The mean of Z i.e. $E[Z]$ is found as:

$$1 \times \pi + 0 \times (1-\pi) = \pi$$

To find the variance, we first need

$$E[Z^2] = 1^2 \times \pi + 0^2 \times 1 - \pi = \pi$$

, so

$$E[Z^2] - E[Z]^2 = \pi - \pi^2 = \pi(1-\pi).$$

1.2. Binomial distribution

By convention we write $X \sim B(n, \pi)$

Suppose we have a sequence of n Bernoulli trials, with Bernoulli random variables $Z_1, Z_2, \ldots, Z_n$ (all taking on values 1 or 0). The random variable $X = Z_1 + Z_2 + \ldots + Z_n$ essentially denotes the number of *successes* amongst the n Bernoulli trials.

This value has a formula as follows:

$$f(x) = p[X = x] = \binom{n}{x} \pi^x (1 - \pi)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

Note that $E[X] = np$ and $V[X] = npq$.

Nowadays, it is pretty straightforward to compute these values, but once upon a time it was conventional to work with tables of probabilities. Let's work one example through.

Begin Quiz Consider a computer network with $n = 5$ computers. The probability that a machine develops a fault in any given week is $\pi = 0.1$. We wish to examine X, the number of computers that are faulty in our sample of 5.

In formal notation, we say that $X \sim Binomial(5, 0.1)$ (or $X \sim B(5, 0.1)$). Find the following:

1. The probability that no computers fail (2d.p.)

2. The probability that one computer fails (2d.p.)

3. The probability that two computers fail (2d.p.)

4. The probability that three computers fail (2d.p.)

5. The probability that four computers fail (2d.p.)

6. The probability that five computers fail (2d.p.)

End Quiz Correct Points: Score:

This density function will be plotted on the next page, along with a $X \sim B(5, 0.5)$, $X \sim B(100, 0.1)$ and $X \sim B(100, 0.5)$.

1.3. The Geometric Distribution

By convention we write $X \sim Go(\pi)$

We shall briefly consider the scenario whereby successive Bernoulli trials are continued until the first success occurs. Let X denote the number of trials up to an including the first success. This gives rise to what we call the Geometric distribution.

Definition 1 *For data assumed to follow the Geometric distribution, X has a probability density function:*

$$p[X = x] = (1 - \pi)^{x-1}\pi; x = 1, 2, \ldots; 0 \leq \pi \leq 1$$

Note that $E[X] = \frac{1}{\pi}$ and $V[X] = \frac{1-\pi}{\pi^2}$.

For example, if a salesperson wishes to calculate the probability that they will make their first successful sale of the day to the fifth customer, given that the probability of a sale $\pi = 0.2$ is given by $p[X = 5] = (1 - 0.2)^{5-1} 0.2 = 0.08192$

We can also define X to be the number of trials preceding the first success, in which case the probability density function is given by:

$$p[X = x] = (1 - \pi)^x \pi$$

For example, a burglar want to know that he will get away with three "jobs" before he finally gets caught. Assuming the probability of being caught (a success for the rest of us) is $\pi = 0.6$ we have $p[X = 3] = (1 - 0.6)^3 0.6 = 0.0384$

1.4. The negative binomial distribution

By convention we would write $X \sim NB(r, \pi)$

If we extend the previous idea so that we can continue the Bernoulli trials until the rth success, and denote X as the number of such trials required, then X has a negative binomial distribution.

Definition 2 *The negative binomial distribution is given by:*

$$p[X = x] = \binom{x-1}{r-1} \pi^r (1-\pi)^{x-r}$$

Note that $E[X] = \frac{r}{\pi}$ and $V[X] = \frac{r(1-\pi)}{\pi^2}$

Consider a country where the prime minister has an interesting private life The probability that a person believes a rumour about this prime minister is $\pi = 0.25$.

- What are the probabilities that the sixth person to hear the rumour will be the sixth to believe it: $p[X = 6|r = 6, \pi = 0.25] = \binom{(6-1)}{(6-1)} 0.25^6 (1 - 0.25)^{6-6}$ which should equal 0.00024
- What is the probability that the twelfth person to hear the rumour will be the fourth to believe it: $p[X = 4|r = 12, \pi = 0.25] = \binom{(12-1)}{(4-1)} 0.25^4 (1 - 0.25)^{12-4}$ which should equal 0.065

1.5. The Poisson distribution

By convention we would write $X \sim P(\lambda)$

The Poisson distribution is used to model events occurring at random in time, area, volume.

Definition 3 *The Poisson distribution, with parameter λ is given as follows:*

$$p[X = x] = \frac{\lambda^x e^{-\lambda}}{x!}; x = 0, 1, 2, \ldots ; 0 \leq \lambda \leq \infty$$

Note that $E[X] = \lambda$ and $V[X] = \lambda$.

The number of accidents on the Pasadena Freeway between 9am and 10am on a Sunday morning is a random variable, thought to follow a Poisson distribution with $\lambda = 7.3$. Find the probability that there will be:

- Exactly 6 accidents.
  For this we need to find $p[X = 6|\lambda = 7.3]$ which is given by $\frac{7.3^6 e^{-7.3}}{6!} = 0.1420$
- Less than 4 accidents.
  For this we need the distribution which is given by $p[X < 4|\lambda = 7.3]$. We find this by taking the following
  $p[X = 0|\lambda = 7.3] + p[X = 1|\lambda = 7.3] + p[X = 2|\lambda = 7.3] + p[X = 3|\lambda = 7.3]$. This is given by calculating
  $\frac{7.3^0 e^{-7.3}}{0!} + \frac{7.3^1 e^{-7.3}}{1!} + \frac{7.3^2 e^{-7.3}}{2!} + \frac{7.3^3 e^{-7.3}}{3!}$ which should give 0.0674

These are the most important examples of discrete probability functions (although there are hundreds available). It is worth being familiar with the key properties of these densities.

Solutions to Quizzes

Solution to Quiz: 0.59

Click on that green button to return to the quiz $\longrightarrow$ ■

Solution to Quiz: 0.33

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Solution to Quiz: 0.07

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Solution to Quiz: 0.01

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Solution to Quiz: 0.00 ■

Solution to Quiz: 0.00

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